Module – I

1.1 WHAT IS A SIGNAL

We are all immersed in a sea of signals. All of us from the smallest living unit, a cell, to the most complex living organism (humans) are all time receiving signals and are processing them. Survival of any living organism depends upon processing the signals appropriately. What is signal? To define this precisely is a difficult task. Anything which carries information is a signal. In this course we will learn some of the mathematical representations of the signals, which has been found very useful in making information processing systems. Examples of signals are human voice, chirping of birds, smoke signals, gestures (sign language), fragrances of the flowers. Many of our body functions are regulated by chemical signals, blind people use sense of touch. Bees communicate by their dancing pattern. Some examples of modern high speed signals are the voltage charger in a telephone wire, the electromagnetic field emanating from a transmitting antenna, variation of light intensity in an optical fiber. Thus we see that there is an almost endless variety of signals and a large number of ways in which signals are carried from on place to another place. In this course we will adopt the following definition for the signal: A signal is a real (or complex) valued function of one or more real variable(s).When the function depends on a single variable, the signal is said to be one dimensional. A speech signal, daily maximum temperature, annual rainfall at a place, are all examples of a one dimensional signal. When the function depends on two or more variables, the signal is said to be multidimensional. An image is representing the two dimensional signal, vertical and horizontal coordinates representing the two dimensions. Our physical world is four dimensional (three spatial and one temporal).

1.2 CLASSIFICATION OF SIGNALS

As mentioned earlier, we will use the term signal to mean a real or complex valued function of real variable(s). Let us denote the signal by $x(t)$. The variable t is called independent variable and the value x of t as dependent variable. We say a signal is continuous time signal if the independent variable t takes values in an interval. For example $t \in (-\infty, \infty)$, $or\ t\in [0, \infty]$ $or\ t \in [T0, T1]$. The independent variable t is referred to as time, even though it may not be actually time. For example in variation if pressure with height t refers above

mean sea level. When t takes vales in a countable set the signal is called a discrete time
signal. For example
$t \epsilon\{0, T, 2T, 3T, 4T, ...\}$ or $t \epsilon\{...-1, 0, 1, ...\}$ or $t \epsilon\{1/2, 3/2, 5/2, 7/2, ...\}$ etc.
For convenience of presentation we use the notation x[n] to denote discrete time signal. Let us pause here and clarify the notation a bit. When we write $x(t)$ it has two meanings. One is value of x at time t and the other is the pairs($x(t)$, t) allowable value of t. By signal we mean the second interpretation. To keep this distinction we will use the following notation: $\{x(t)\}$ to denote the continuous time signal. Here $\{x(t)\}$ is short notation for $\{x(t), t \epsilon I\}$ where I is the set in which t takes the value. Similarly for discrete time signal we will use the notation $\{x[n]\}$, where $\{x[n]\}$ is short for $\{x[n], n\_I\}$. Note that in $\{x(t)\}$ and $\{x[n]\}$ are dummy variables i.e. $\{x[n]\}$ and $\{x[t]\}$ refer to the same signal. Some books use the notation $x[\cdot]$ to denote $\{x[n]\}$ and $x[n]$ to denote value of x at time $n \cdot x[n]$ refers to the whole waveform, while $x[n]$ refers to a particular value. Most of the books do not make this distinction clean and use $x[n]$ to denote signal and $x[n]$ to denote a particular value.

As with independent variable t, the dependent variable x can take values in a continues set or in a countable set. When both the dependent and independent variable take value in intervals, the signal is called an analog signal. When both the dependent and independent variables take values in countable sets (two sets can be quite different) the signal is called Digital signal. When we use digital computers to do processing we are doing digital signal processing. But most of the theory is for discrete time signal processing where default variable is continuous. This is because of the mathematical simplicity of discrete time signal processing. Also digital signal processing tries to implement this as closely as possible. Thus what we study is mostly discrete time signal processing and what is really implemented is digital signal processing.

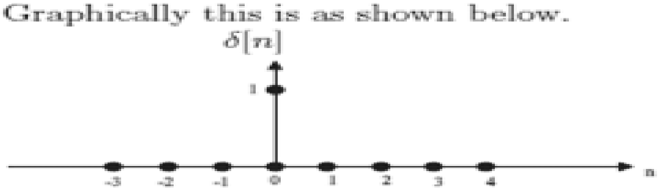
1.3 ELEMENTARY SIGNALS

There are several elementary signals that feature prominently in the study of digital signals and digital signal processing.

(a)Unit sample sequence $\delta[n]$: Unit sample sequence is defined by

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Graphically this is as shown below.



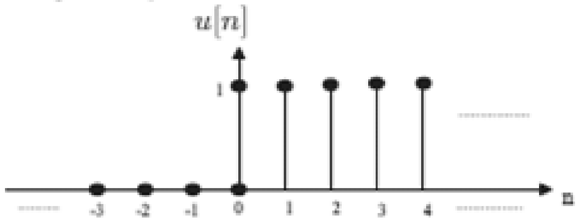
Unit sample sequence is also known as impulse sequence. This plays role akin to the impulse function $\delta(t)$ of continues time. The continues time impulse $\delta(t)$ is purely a mathematical construct while in discrete time we can actually generate the impulse sequence.

(b)Unit step sequence $u[n]$: Unit step sequence is defined by

(b)Unit step sequence $u[n]$: Unit step sequence is defined by

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

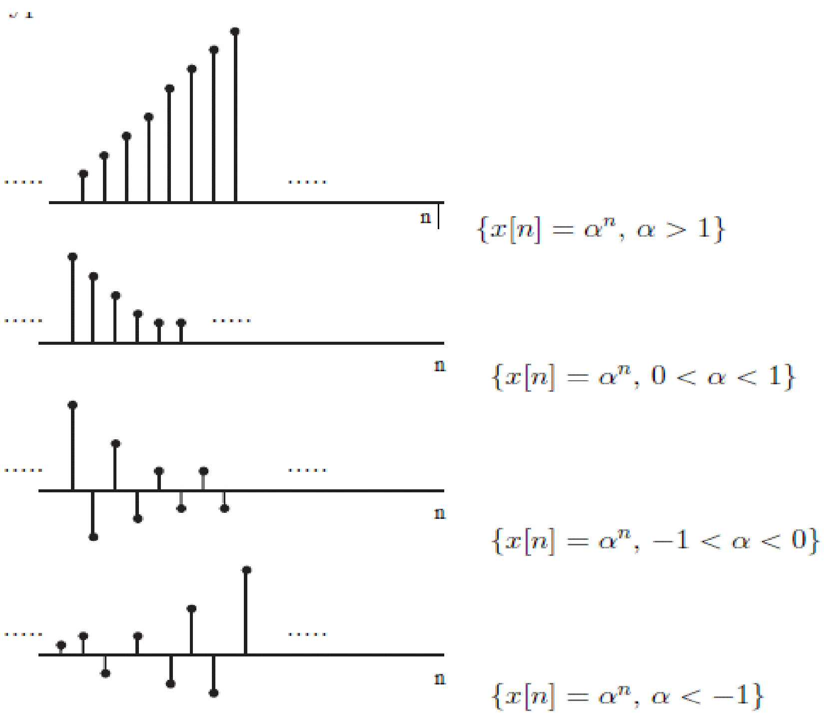
Graphically this is as shown below



(c) Exponential sequence: The complex exponential signal or sequence $x[n]$ is defined by $x[n] = C\,\alpha^n$

where C and α are, in general, complex numbers.

Real exponential signals: If C and α are real, we can have one of the several type of behaviour illustrated below



2. SIMPLE OPERATIONS AND PROPERTIES OF SEQUENCES

2.1 Simple operations on signals

In analyzing discrete-time systems, operations on sequences occur frequently. Some operations are discussed below.

2.1.1 Sequence addition:

Let $\{x[n]\}$ and $\{y[n]\}$ be two sequences. The sequence addition is defined as term by term addition. Let $\{z[n]\}$ be the resulting sequence

$\{z[n]\} = \{x[n]\} + \{y[n]\}$, where each term $z[n] = x[n] + y[n]$

We will use the following notation

$\{x[n]\} + \{y[n]\} = \{x[n] + y[n]\}$

2.1.2 Scalar multiplication:

Let a be a scalar. We will take a to be real if we consider only the real valued signals, and take a to be a complex number if we are considering complex valued sequence. Unless otherwise stated we will consider complex valued sequences. Let the resulting sequence be denoted by $w[n]$

$\{w[n]\} = ax[n]$ is defined by $w[n] = ax[n]$, each term is multiplied by a

We will use the notation $aw[n] = aw[n]$

Note: If we take the set of sequences and define these two operators as addition and scalar multiplication they satisfy all the properties of a linear vector space.

2.1.3 Sequence multiplication:

Let $\{x[n]\}$ and $\{y[n]\}$ be two sequences, and $\{z[n]\}$ be resulting sequence

$\{z[n]\} = \{x[n]\}\{y[n]\}$, where $z[n] = x[n]y[n]$. The notation used for this will be $\{x[n]\}\{y[n]\} = \{x[n]y[n]\}$

Now we consider some operations based on independent variable n.

2.1.4 Shifting

This is also known as translation. Let us shift a sequence $\{x[n]\}$ by $n0$ units, and the resulting sequence by $\{y[n]\}$

$\{y[n]\} = z{-}n0(\{x[n]\})$

where $z{-}n0()$is the operation of shifting the sequence right by $n0$ unit. The terms are defined by $y[n] = x[n{-}n)]$. We will use short notation $\{x[n{-}n0]\}$

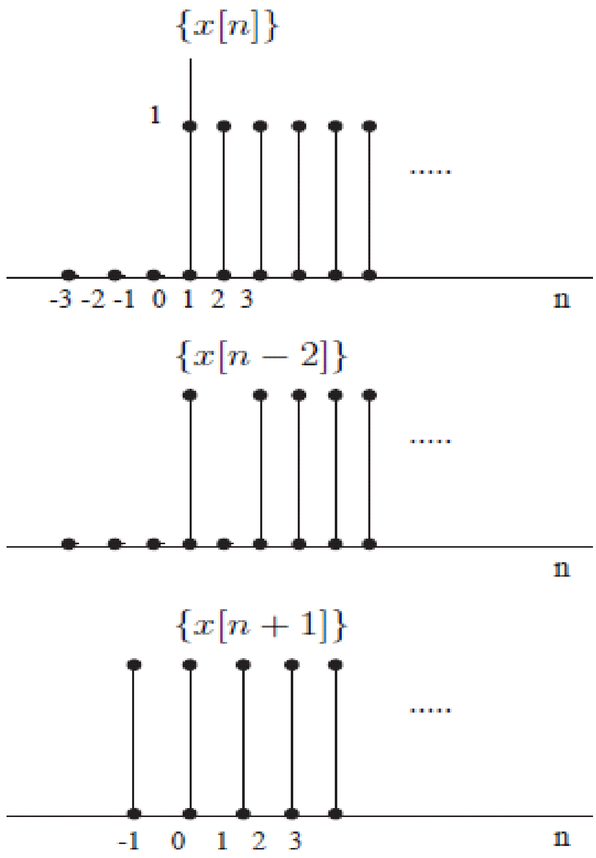
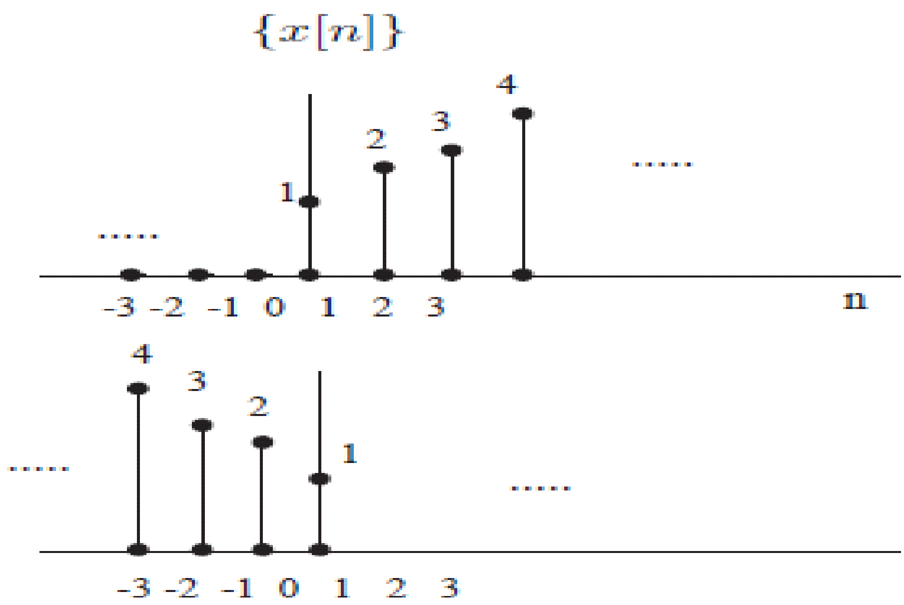


Figure above show some examples of shifting. A negative value of n_0 shift towards right.

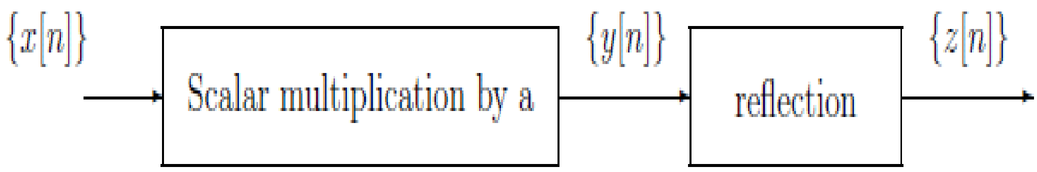
2.1.5 Reflection:

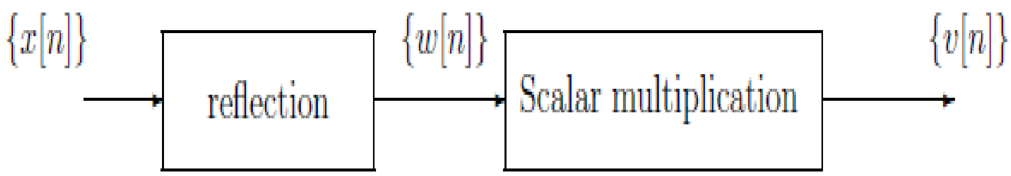
Let $\{x[n]\}$ be the original sequence, and $\{y[n]\}$ be reflected sequence, then $y[n]$ is defined by $y[n] = x[-n]$

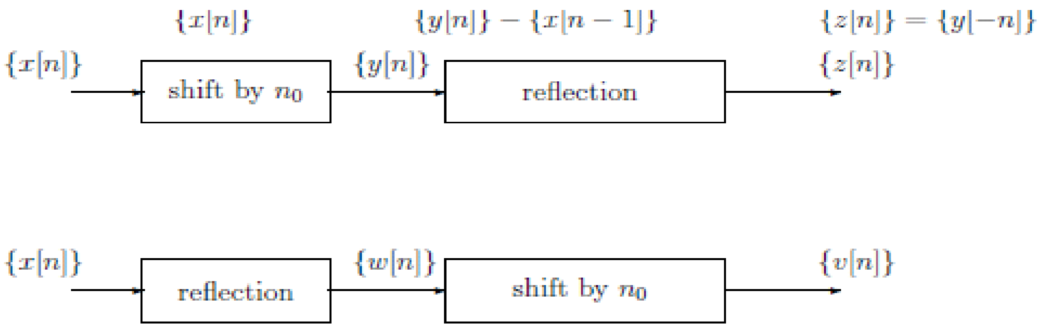


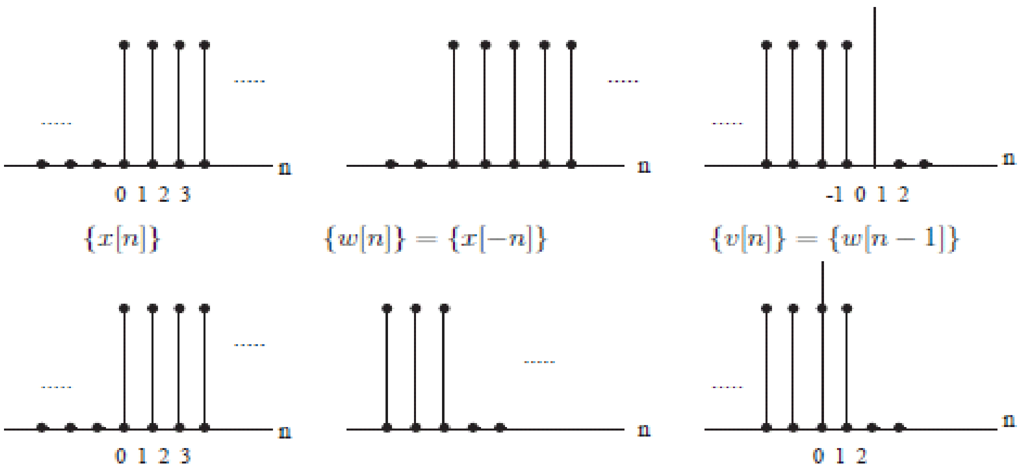
We will denote this by $\{x[n]\}$. When we have complex valued signals, sometimes we reflect and do the complex conjugation, ie, $y[n]$ is defined by $y[n] = x^*[-n]$, where * denotes complex conjugation. This sequence will be denoted by $\{x^*[-n]\}$.

We will learn about more complex operations later on. Some of these operations commute, i.e. if we apply two operations we can interchange their order and some do not commute. For example scalar multiplication and reflection.









We can combine many of these operations in one step, for example $\{y[n]\}$ may be defined as $y[n] = 2x[3 - n]$.

2.2 SOME PROPERTIES OF SIGNALS:

2.2.1 Energy of a Signal:

The total enery of a signal *{x[n]}* is defined by

$$E_x = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

A signal is reffered to as an energy signal, if and only if the total energy of the signal *Ex* is finite. An energy signal has a zero power and a power signal has infinite energy. There are signals which are neither energy signals nor power signals. For example *{x[n]}* defined by *x*[*n*] = *n* does not have finite power or energy

2.2.2 Power of a signal:

If *{x[n]}* is a signal whose energy is not finite, we define power of the signal

$$y(n) = -\sum_{i=1}^{N} a_i y(n-i) + \sum_{j=0}^{M} b_j x(n-j).$$

$$P_x = \lim_{N\to\infty} \frac{1}{(2N+1)} \sum_{n=-N}^{N} |x[n]|^2$$

A signal is referred to as a power signal if the power P_x satisfies the condition

$$0 < P_x < \infty$$

2.2.3 Periodic Signals:

An important class of signals that we encounter frequently is the class of periodic signals. We say that a signal $\{x[n]\}$ is periodic period N, where N is a positive integer, if the signal is unchanged by the time shift of N ie.,

$$\{x[n]\} = \{x[n+N]\}$$

or $x[n] = x[n+N$ for all n.

Since $\{x[n]\}$ is same as $\{x[n+N]\}$, it is also periodic so we get

$$\{x[n]\} = \{x[n+N]\} = \{x[n+N+N]\} = \{x[n+2N]\}$$

Generalizing this we get $\{x[n]\} = \{x[n+kN]\}$, where k is a positive integer. From this we see that $\{x[n]\}$ is periodic with $2N$, $3N$, ... The fundamental period N_0 is the smallest positive value N for which the signal is periodic. The signal illustrated below is periodic with fundamental period $N_0 = 4$. $\{x[n]\}$ By change of variable we can write $\{x[n]\} = \{x[n+N]\}$ as $\{x[m-N]\} = \{x[m]\}$ and then we see that

$$\{x[n]\} = \{x[n+kN]\},$$

for all integer values of k, positive, negative or zero. By definition, period of a signal is always a positive integer n. Except for a all zero signal all periodic signals have infinite energy. They may have finite power. Let $\{x[n]\}$ be periodic with period N, then the power P_x is given by

$$P-x=\lim_{M\to\infty}\frac{1}{(2M+1)}\sum_{n=-M}^{M}|x[n]|^2$$

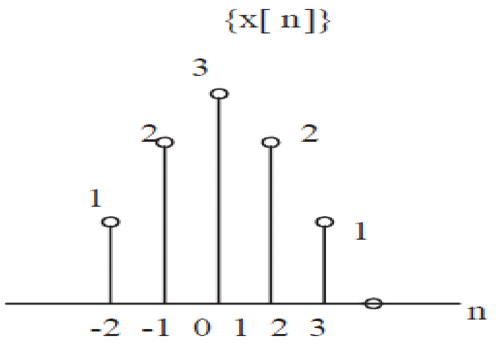
$$=\lim_{M\to\infty}\tfrac{1}{2M+1}[\sum_{n=0}^{N-1}|x[n]|^2+\sum_{n=N}^{2N-1}|x[n]|^2+\ldots$$
$$+\sum_{n=(k-1)N-1}^{kN-1}|x[n]|^2+\sum_{n=kN}^{M}|x[n]|^2+\sum_{n=-N}^{-1}|x[n]|^2+\ldots$$
$$+\sum_{n=-kN}^{-(k-1)N-1}|x[n]|^2+\sum_{n=-M}^{-kN-1}|x[n]|^2]$$

where k is largest integer such that $kN-1\leq M$. Since the signal is periodic, sum over one period will be same for all terms. We see that k is approximately equal to M/N(it is integer part of this) and for large M we get $2M/N$ terms and limit $2M/(2M+1)$ as M goes to infinite is one we get

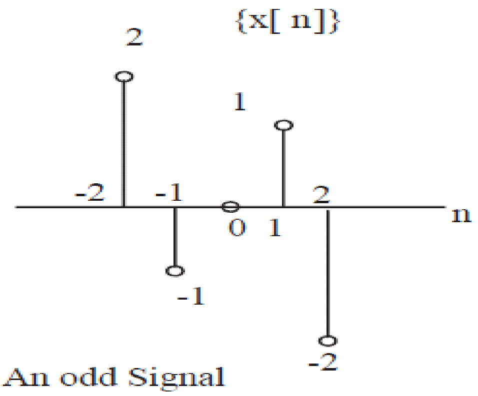
$$P_x=\frac{1}{N}\sum_{n=0}^{N-1}|x[n]|^2$$

2.2.4 Even and odd signals:

A real valued signal $\{x[n]\}$ is referred as an even signal if it is identical to its time reversed counterpart ie, if $\{x[n]\}=\{x[-n]\}$ A real signal is referred to as an odd signal if $\{x[n]\}=\{-x[-n]\}$ An odd signal has value 0 at $n=0$ as $x[0]=-x[n]=-x[0]$



An even signal



An odd Signal

as a sum of an even signal and an odd signal. Consider the signals

$$Ev(\{x[n]\})=\{x_e[n]\}=\{1/2(x[n]+x[-n])\}$$

and

$$Od(\{x[n]\})=\{x_0[n]\}=\{1/2(x[n]-x[-n])\}$$

We can see easily that

$$\{x[n]\}=\{x_e[n]\}+\{x_o[n]\}$$

The signal $\{x[n]\}$ is called the even part of $\{x[n]\}$. We can verify very easily that $\{x_e[n]\}$ is an even signal. Similarly, $\{x_0[n]\}$ is called the odd part of $\{x[n]\}$ and is an odd signal. When we have complex valued signals we use a slightly different terminology. A complex valued signal $\{x[n]\}$ is referred to as a conjugate symmetric signal if $\{x[n]\} = \{x^*[-n]$, where x^*refers to the complex conjugate of x. Here we do reflection and complex conjugation. If $\{x[n]\}$ is real valued this is same as an even signal. A complex signal $\{x[n]\}$ is referred to as a conjugate antisymmetric signal if $\{x[n]\} = \{-x^*[-n]\}$. We can express any complex valued signal as sum conjugate symmetric and conjugate antisymmetric signals. We use notation similar to above $Ev(\{x[n]\}) = \{x_e[n]\} = \{1/2(x[n] + x^*[-n])\}$ and $Od(\{x[n]\}) = \{x_0[n]\} = \{1/2(x[n] - x* [-n])\}$ then $\{x[n]\} = \{x_e[n]\} + \{x_o[n]\}$. We can see easily that $\{xe[n]\}$ is conjugate symmetric signal and $\{x_o[n]\}$ is conjugate antisymmetric signal. These definitions reduce to even and odd signals in case signalstakes only real values.

2.3 PERIODICITY PROPERTIES OF SINUSOIDAL SIGNALS

Let us consider the signal $\{x[n]\} = \{\cos w_0 n\}$. We see that if we replace $w0$ by $(w_0 + 2\pi)$ we get the same signal. In fact the signal with frequency $w_0\}2\pi$,$w0\}4\pi$ and so on. This situation is quite different from continuous time signal $\{\cos w_0 t, -\infty < t < \infty\}$ where each frequency is different. Thus in discrete time we need to consider frequency interval of length 2π only. As we increase $w;$ 0 to π signal oscillates more and more rapidly. But if we further increase frequency from π to 2π the rate of oscillations decreases. This can be seen easily by plotting signal $\cos w_0 n\}$ for several values of w_0. The signal $\{\cos w_0 n\}$ is not periodic for every value of w_0. For the signal to be periodic with period $N > 0$, we should have

$$\{\cos w_0 n\} = \{cos w_0(n + N)\}$$

that is $w_0 N$ should be some multiple of 2π.

$$w_0 N = 2\pi m$$

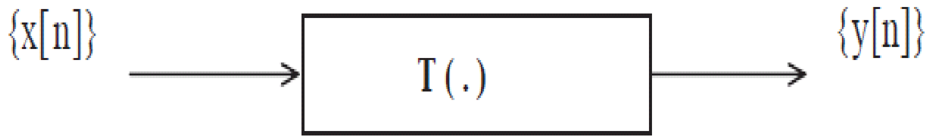
or

$$\frac{w_0}{2\pi} = \frac{m}{N}$$

Thus signal $\{\cos w_0 n\}$ is periodic if and only if $w_0{=}2\pi$ is a rational number.
Above observations also hold for complex exponential signal $\{x[n]\} = \{e^{jw_0 n}\}$

2.3.1.Discrete-Time Systems

A discrete-time system can be thought of as a transformation or operator that maps an input sequence $\{x[n]\}$ to an output sequence $\{y[n]\}$



By placing various conditions on $T(\cdot)$ we can define different classes of systems.

3.BASIC SYSTEM PROPERTIES

3.1 Systems with or without memory:

A system is said to be memory less if the out put for each value of the independent variable at a given time n depends only on the input value at time n. For example system specified by the relationship $y[n] = \cos(x[n]) + z$ is memory less. A particularly simple memory less system is the identity system defined by $y[n] = x[n]$ In general we can write input-output relationship for memory less system as $y[n] = g(x[n])$. Not all systems are memory less. A simple example of system with memory is a delay defined by $y[n] = x[n - 1]$
A system with memory retains or stores information about input values at times other than the current input value.

3.2 Inevitability

A system is said to be invertible if the input signal $\{x[n]\}$ can be recovered from the output signal $\{y[n]\}$. For this to be true two different input signals should produce two different outputs. If some different input signal produce same output signal then by processing output we can not say which input produced the output. Example of an invertible system is

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

then

$$x[n] = y[n] - y[n-1]$$

Example if a non-invertible system is

$$y[n] = 0$$

That is the system produces an all zero sequence for any input sequence. Since every input sequence gives all zero sequence, we can not find out which input produced the output. The system which produces the sequence $\{x[n]\}$ from sequence $\{y[n]\}$ is called the inverse system. In communication system, decoder is an inverse of the encoder.

3.3 Causality

A system is causal if the output at anytime depends only on values of the input at the present time and in the past. $y[n] = f(x[n], x[n-1], ...)$. All memory less systems are causal. An accumulator system defined by

$$y[n] = \sum_{x[h]}$$

is also causal. The system defined by

$$y[n] = \frac{1}{2N+1} \sum_{k=-N}^{N} x[n-k]$$

For real time system where n actually denoted time causalities is important. Causality is not an essential constraint in applications where n is not time, for example, image processing. If we case doing processing on recorded data, then also causality may not be required.

3.4 Stability

There are several definitions for stability. Here we will consider bounded input bonded output(BIBO) stability. A system is said to be BIBO stable if every bounded input produces a bounded output. We say that a signal $\{x[n]\}$ is bounded if

$$|x[n]| < M < \infty \qquad \text{for all } n$$

The moving averafe system

$$y[n] = \frac{1}{2N+1} \sum_{k=-N}^{N} x[n]$$

is stable as $y[n]$ is sum of finite numbers and so it is bounded. The accumulator system defined by

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

is unstable. If we take $\{x[n]\} = \{u[n]\}$, the unit step then $y[0] = 1$, $y[1] = 2$, $y[2] = 3$, are$y[n] = n + 1$, $n \geq 0$ so $y[n]$ grows without bound.

3.5 Time invariance

A system is said to be time invariant if the behaviour and characteristics of the system do not change with time. Thus a system is said to be time invariant if a time delay or time advance in the input signal leads to identical delay or advance in the output signal. Mathematically if

$$\{y[n]\} = T(\{x[n]\})$$

then

$$\{y[n-n_0]\} = T(\{x[n-n_0]\}) \text{ for any } n_0$$

Let us consider the accumulator system

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

If the input is now $\{x_1[n]\} = \{x[n-n_0]\}$ then the corresponding output is

$$y_1[n] = \sum_{k=-\infty}^{n} x_1[k]$$

$$= \sum_{k=-\infty} x[k]$$

The shifted output signal is given by

$$y[n-n_0] == \sum_{k=-\infty}^{n-n_0} x[k]$$

The two expression look different, but infact they are equal. Let us change the index of summation by $l = k - n_0$ in the first sum then we see that

$$y_1[n] == \sum_{l=-\infty}^{n-n_0} x[l]$$

$$= y[n-n_0]$$

Hence, $\{y[n]\} = \{y[n-n_0]\}$ and the system is time-invariant. As a second example consider the system defined by

$$y[n] = nx[n]$$

if

$$\{x_1[n]\} = \{x[n-n_0]\}$$

$$y_1[n] = nx_1[n] = nx[n-n_0]$$

while

$$y[n-n_0] = (n-n_0)x[n-n_0]$$

and so the system is not time-invariant. It is time varying. We can also see this by giving a counter example. Suppose input is $\{x[n]\} = \{\delta[n]\}$ then output is all zero sequence. If the input is $\{\delta[n-1]\}$ then output is $\{\delta[n-1]\}$ which is definitely not a shifted version version of all zero sequence.

3.6 Linearity

This is an important property of the system. We will see later that if we have system which is linear and time invariant then it has a very compact representation. A linear system possesses the important property of super position: if an input consists of weighted sum of several signals, the output is also weighted sum of the responses of the system to each of those input signals. Mathematically let $\{y_1[n]\}$ be the response of the system to the input $\{x_1[n]\}$ and let $\{y_2[n]\}$ be the response of the system to the input $\{x_2[n]\}$. Then the system is linear if:

1. **Additivity**: The response to $\{x_1[n]\} + \{x_2[n]\}$ is $\{y_1[n]\} + \{y_2[n]\}$
2. **Homogeneity**: The response to $a\{x_1[n]\}$ is $a\{y_1[n]\}$, where a is any real number if we are considering only real signals and a is any complex number if we are considering complex valued signals.
3. **Continuity:** Let us consider $\{x_1[n]\}, \{x_2[n]\}, \ldots\{x_k[n]\}\ldots$ be countably infinite number of signals such that

$\lim_{k\to\infty}\{ x_k[n]\} = \{x[n]\}$ Let the corresponding output signals be denoted by $\{yn[n]\}$

and $\lim_{k\to\infty} \{ yn[n]\} =\{y[n]\}$ We say that system processes the continuity property

if the response of the system to the limiting input $\{x[n]\}$ is limit of the responses $\{y[n]\}$.

$$T\left(\lim_{k\to\infty}\{ xk[n]\}\right) = \lim_{k\to\infty} T(\{Xk[n]\})$$

The additive and continuity properties can be replaced by requiring that We say that system posseses the continuity property system is additive for countably infinite number if signals i.e. response to$\{x1[n]\}+\{x2[n]\}+\ldots+\{xn[n]\}+\ldots$ is $\{y1[n]\}+\{y2[n]\}+\ldots+\{yk[n]\}+\ldots$.Most of the books do not mention the continuity property. They state only finite additivity and homogeneity. But from finite additivity we can not deduce c....... additivity. This distinction becomes very important in continuous time systems. A system can be linear without being time invariant and it can be time invariant without being linear. If a system is linear, an all zero input sequence will produce a all zero output sequence. Let

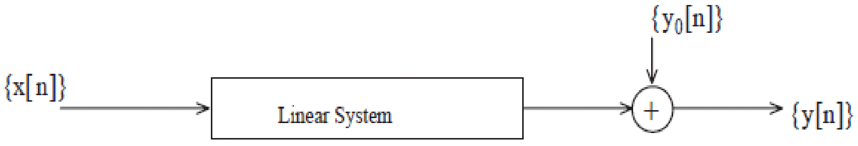
$\{0\}$ denote the all zero sequence ,then $\{0\} = 0.\{x[n]\}$. If $T(\{x[n]\} = \{y[n]\})$ then by homogeneity property $T(0.\{x[n]\}) = 0.\{y[n]\}$

$$T(\{0\}) = \{0\}$$

Consider the system defined by

$$y[n] = 2x[n] + 3$$

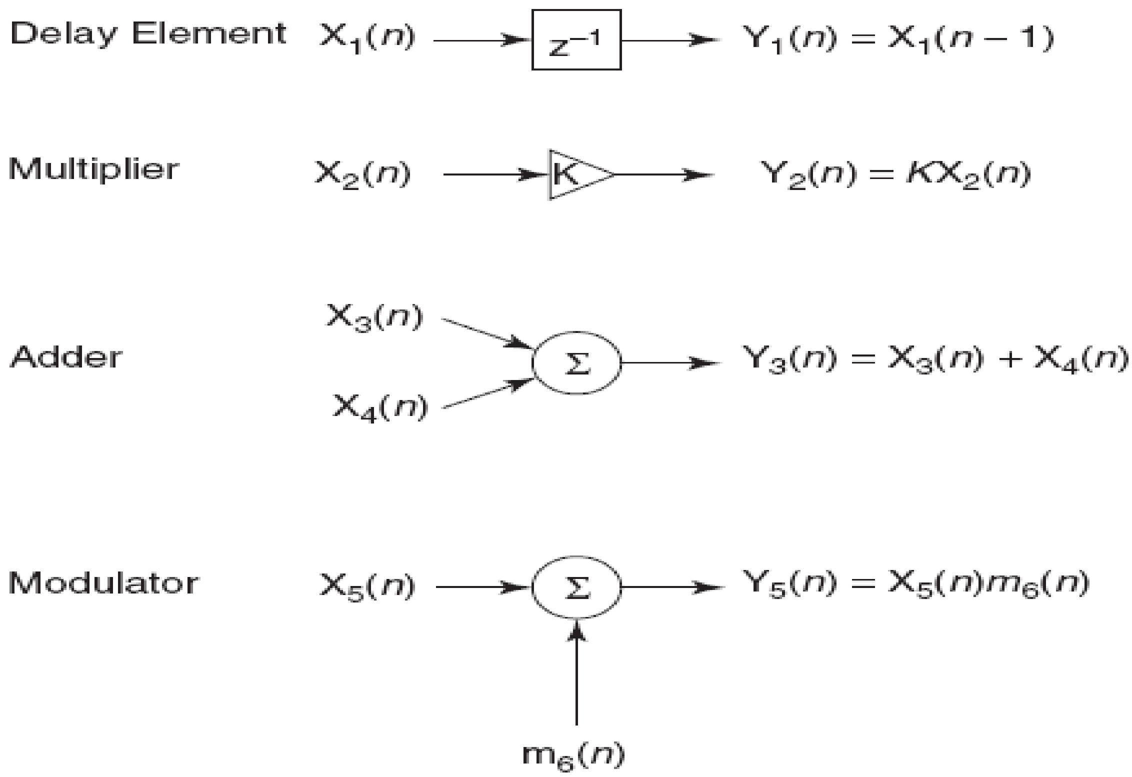
This system is not linear. This can be verified in several ways. If the input is all zero sequence $\{0\}$, the output is not an all zero sequence. Although the defining equation is a linear equation is x and y the system is nonlinear. The output of this system can be represented as sum of a linear system and another signal equal to the zero input response. In this case the linear system is $y[n] = 2x[n]$ and the zero-input response is $y_0[n] = 3$ for all n



systems correspond to the class of incrementally linear system. System is linear in term of difference signal i.e if we define $\{x_d[n]\} = \{x_1[n]\} - \{X_2[n]\}$and $\{y_d[n]\} = \{y_1[n]\} - \{y_2[n]\}$. Then in terms of $\{x_d[n]\}$ and $\{y_d[n]\}$ the system is linear.

4. MODELS OF THE DISCRETE-TIME SYSTEM

First let us consider a discrete-time system as an interconnection of only three basic components: the delay elements, multipliers, and adders. The input–output relationships for these components and their symbols are shown in Figure below. The fourth component is the modulator, which multiplies two or more signals and hence performs a nonlinear operation.



The *basic components used in a discrete-time system.*

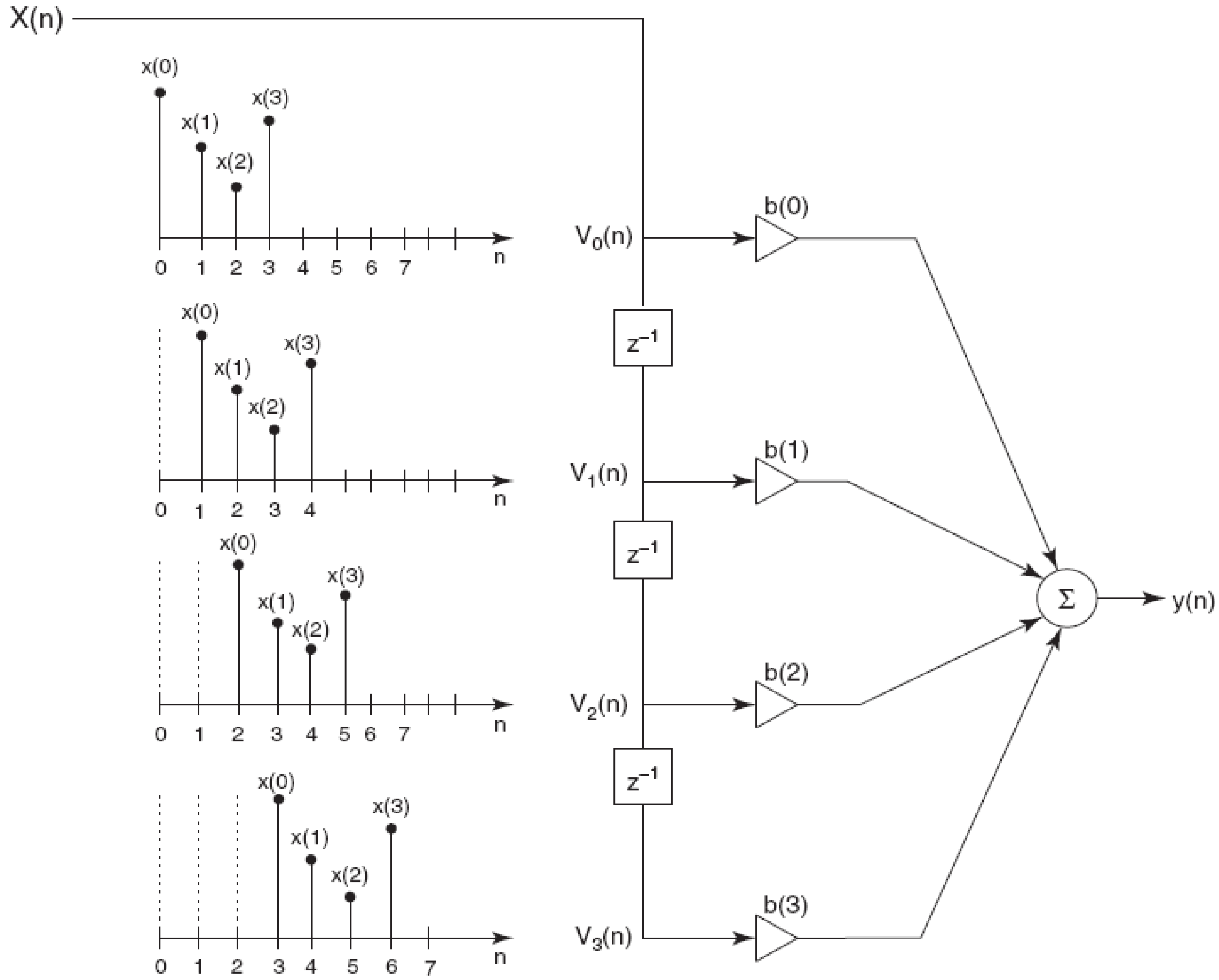
A simple discrete-time system is shown in Figure 5, where input signal $x(n)$= {$x(0)$, $x(1)$, $x(2)$, $x(3)$} is shown to the left of $v_0(n)$= $x(n)$. The signal $v_1(n)$shown on the left is the signal $x(n)$delayed by T seconds or one sample, so, $v_1(n)$= $x(n-1)$. Similarly, $v(2)$and $v(3)$are the signals obtained from $x(n)$when it is delayed by $2T$ and $3T$ seconds: $v_2(n)$= $x(n-2)$and $v_3(n)$= $x(n-3)$. When we say that the signal $x(n)$is delayed by T, $2T$, or $3T$ seconds, we mean that the samples of the sequence are present T, $2T$, or $3T$ seconds *later*, as shown by the plots of the signals to the left of $v_1(n)$, $v_2(n)$, and $v_3(n)$. But at any given time $t = nT$, the samples in $v_1(n)$, $v_2(n)$, and $v_3(n)$ are the samples of the input signal that occur T, $2T$, and $3T$ seconds *previous* to $t= nT$. For example, at $t = 3T$, the value of the sample in $x(n)$is $x(3)$, and the values present in $v_1(n)$, $v_2(n)$and $v_3(n)$are $x(2)$, $x(1)$, and $x(0)$, respectively.

A good understanding of the operation of the discrete-time system as illustrated in above Figure is essential in analyzing, testing, and debugging the operation of the system when available software is used for the design, simulation, and hardware implementation of the system.

It is easily seen that the output signal in above Figure is

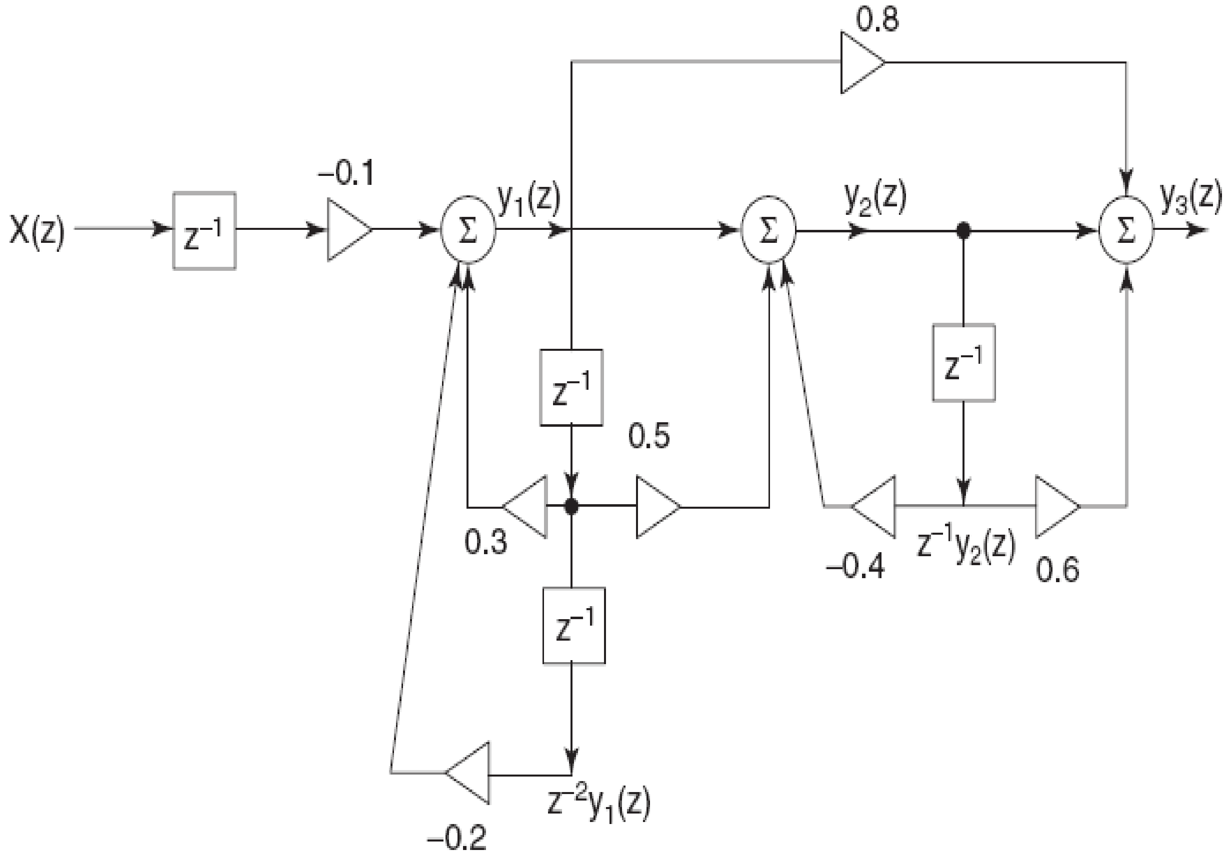
$$y(n) = b(0)v(0) + b(1)v(1) + b(2)v(2) + b(3)v(3)$$
$$= b(0)x(n) + b(1)x(n-1) + b(2)x(n-2) + b(3)x(n-3)$$

where $b(0)$, $b(1)$, $b(2)$, $b(3)$are the gain constants of the multipliers. It is also easy to see from the last expression that the output signal is the weighted sum of the current value and the previous three values of the input signal. So this gives us an input–output relationship for the system shown in below



Operations in a typical discrete-time system.

Now we consider another example of a discrete-time system, shown in Figure 5. Note that a fundamental rule is to express the output of the adders and generate as many equations as the number of adders found in this circuit diagram for the discrete-time system. (This step is similar to writing the node equations for an analog electric circuit.) Denoting the outputs of the three adders as $y_1(n)$, $y_2(n)$, and $y_3(n)$, we get



Schematic circuit for a discrete-time system.

$$y_1(n) = 0.3y_1(n-1) - 0.2y_1(n-2) - 0.1x(n-1)$$

$$y_2(n) = y_1(n) + 0.5y_1(n-1) - 0.4y_2(n-1)$$

$$y_3(n) = y_2(n) + 0.6y_2(n-1) + 0.8y_1(n)$$

These three equations give us a mathematical model derived from the model shown in above that is schematic in nature. We can also derive (draw the circuit realization) the model shown in Figure 5 from the same equations given above. After eliminating the internal variables $y_1(n)$and $y_2(n)$; that relationship constitutes the third model for the system. The general form of such an input–output relationship is

$$y(n) = -\sum_{k=1}^{N} a(k)y(n-k) + \sum_{k=0}^{M} b(k)x(n-k) \qquad \text{Eq(1)}$$

or in another equivalent form

$$\sum_{k=0}^{N} a(k)y(n-k) = \sum_{k=0}^{M} b(k)x(n-k); \qquad a(0) = 1 \quad \text{Eq(2)}$$

Eq(1) shows that the output $y(n)$is determined by the weighted sum of the previous N values of the output and the weighted sum of the current and previous M + 1 values of the input. Very often the coefficient $a(0)$as shown in Eq(2) is normalized to unity.

5. LINEAR TIME-INVARIANT, CAUSAL SYSTEMS

In this section, we study linear time-invariant causal systems and focus on properties such as linearity, time invariance, and causality.

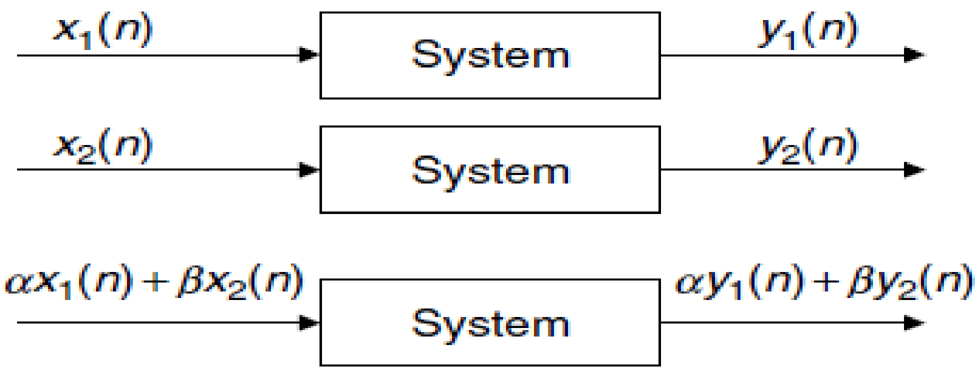
5.1 Linearity:

A linear system is illustrated in below figure, where $y_1(n)$ is the system output using an input $x_1(n)$, and $y_2(n)$ is the system output using an input $x_2(n)$. This Figure illustrates that the system output due to the weighted sum inputs $\alpha x_1(n) + \beta x_2(n)$ is equal to the same weighted sum of the individual outputs obtained from their corresponding inputs, that is

$$y(n)=\alpha y_1(n) + \beta y_2(n)$$

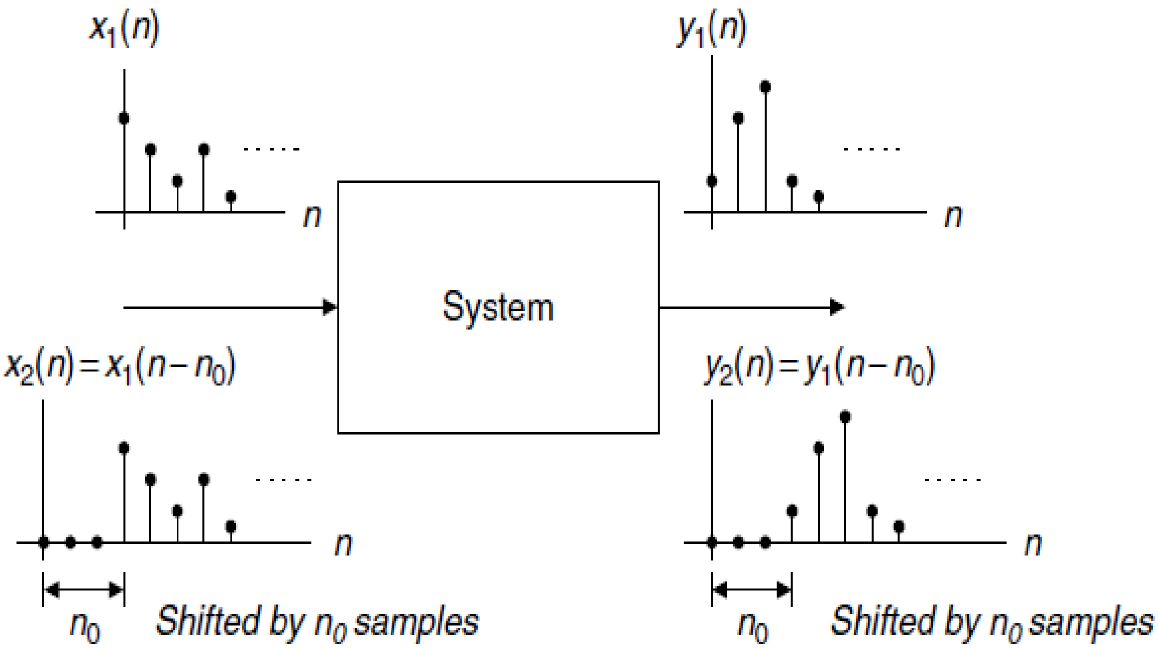
where α and β are constants.

For example, assuming a digital amplifier as $y(n)=10x(n)$, the input is multiplied by 10 to generate the output. The inputs $x_1(n) = u(n)$ and $x_2(n) =\delta(n)$ generate the outputs $y_1(n) =10u(n)$ and $y_2(n) = 10\delta(n)$, respectively. If, as described in below Figure , we apply to the system using the combined input $x(n)$, where the first input is multiplied by a constant 2 while the second input is multiplied by a constant 4, $x(n) = 2x_1(n) + 4x_2(n) =2u(n) + 4\delta(n)$,



5.2 Time Invariance

A time-invariant system is illustrated in Figure below, where $y_1(n)$ is the system output for the input $x_1(n)$. Let $x2(n) = x_1(n - n0)$ be the shifted version of $x_1(n)$



by n_0 samples. The output $y_2(n)$ obtained with the shifted input $x2(n) = x1(n - n_0)$is equivalent to the output $y_2(n)$ acquired by shifting $y_1(n)$ by n_0 samples,$y_2(n) = y_1(n - n_0)$.This can simply be viewed as the following. If the system is time invariant and $y_1(n)$ is the system output due to the input $x_1(n)$,then the shifted system input $x_1(n - n_0)$ will produce a shifted system output $y_1(n - n_0)$by the same amount of time n_0.

5.3 Differential Equations and Impulse Responses:

A causal, linear, time-invariant system can be described by a difference equation having the following general form:

$y(n) + a_1y(n - 1) + \dots + a_Ny(n - N) = b_0x(n) + b_1x(n - 1) + \dots + b_Mx(n - M)$

where $a_1, \dots, a_N$ and $b_0, b_1, \dots, b_M$ are the coefficients of the difference equation. It can further be written as

$y(n) = - a_1y(n - 1) - \dots - a_Ny(n - N) + b_0x(n) + b_1x(n - 1) + \dots + b_Mx(n - M)$

6. FOURIER SERIES COEFFICIENTS OF PERIODIC IN DIGITAL SIGNALS:

Let us look at a process in which we want to estimate the spectrum of a periodic digital signal x(n) sampled at a rate of fs Hz with the fundamental period T_0 =

NT, as shown in below, where there are N samples within the duration of the fundamental period and T = 1/fs is the sampling period. For the time being, we assume that the periodic digital signal is band limited to have all harmonic frequencies less than the folding frequency fs=2 so that aliasing does not occur. According to Fourier series analysis (Appendix B), the coefficients of the Fourier series expansion of a periodic signal x(t) in a complex form is

$$c_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}\,dt \qquad -\infty < k < \infty, \tag{4.1}$$

where k is the number of harmonics corresponding to the harmonic frequency of kf_0 and $\omega_0 = 2\pi/T_0$ and $f_0 = 1/T_0$ are the fundamental frequency in radians per second and the fundamental frequency in Hz, respectively. To apply Equation (4.1), we substitute $T_0 = NT$, $\omega_0 = 2\pi/T_0$ and approximate the integration over one period using a summation by substituting $dt = T$ and $t = nT$. We obtain

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}, \qquad -\infty < k < \infty. \tag{4.2}$$

Since the coefficients c_k are obtained from the Fourier series expansion in the complex form, the resultant spectrum c_k will have two sides. There is an important feature of Equation (4.2) in which the Fourier series coefficient c_k is periodic of N. We can verify this as follows

$$c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi(k+N)n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}e^{-j2\pi n}. \tag{4.3}$$

Since $e^{-j2\pi n} = \cos(2\pi n) - j\sin(2\pi n) = 1$, it follows that

$$c_{k+N} = c_k. \tag{4.4}$$

Therefore, the two-sided line amplitude spectrum jckj is periodic, as shown in Figure 4.3. We note the following points:

a. As displayed in Figure 4.3, only the line spectral portion between the frequency fs=2 and frequency fs=2 (folding frequency) represents the frequency information of the periodic signal.

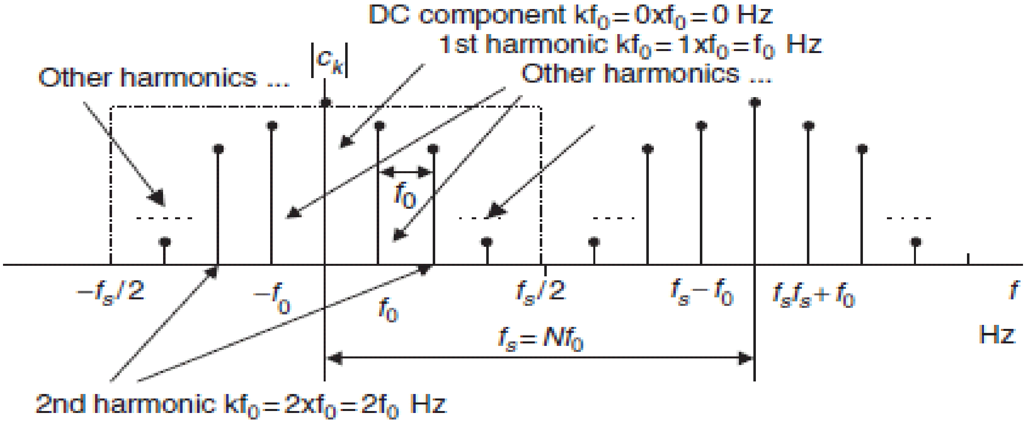


FIGURE 4.3 **Amplitude spectrum of the periodic digital signal.**

b. Notice that the spectral portion from fs=2 to fs is a copy of the spectrum in the negative frequency range from _fs=2 to 0 Hz due to the spectrum being periodic for every Nf_0 Hz. Again, the amplitude spectral components indexed from fs=2 to fs can be folded at the folding frequency fs=2 to match the amplitude spectral components indexed from 0 to fs=2 in terms of fs _ f Hz, where f is in the range from fs=2 to fs. For convenience, we compute the spectrum over the range from 0 to fs Hz with nonnegative indices, that is,

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}},\ k = 0, 1, \ldots,\ N-1. \qquad (4.5)$$

c. For the kth harmonic, the frequency is $f = kf_0$ Hz. The frequency spacing between the consecutive spectral lines, called the frequency resolution, is f_0 Hz

7. Discrete Fourier Transform

Now, let us concentrate on development of the DFT. In below Figure shows one way to obtain the DFT formula. First, we assume that the process acquires data samples from digitizing the interested continuous signal for a duration of T seconds. Next, we assume that a periodic signal x(n) is obtained by copying the acquired N data samples with the duration of T to itself repetitively. Note that we assume continuity between the N data sample frames. This is not true in practice. We will tackle this problem in Section 4.3. We determine the Fourier series coefficients using one-period N data samples and Equation (4.5). Then we multiply the Fourier series coefficients by a factor of N to obtain

$$X(k) = Nc_k = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi kn}{N}}, \quad k = 0,\ 1, \ldots,\ N-1,$$

where X(k) constitutes the DFT coefficients. Notice that the factor of N is a constant and does not affect the relative magnitudes of the DFT coefficients X(k). As shown in the last plot, applying DFT with N data samples of x(n) sampled at a rate of fs (sampling period is T = 1/fs) produces N complex DFT

Using periodicity, it follows that

$$c_{-1} = c_3 = j0.5, \text{ and } c_{-2} = c_2 = 0.$$

b. The amplitude spectrum for the digital signal is sketched in Figure 4.5.

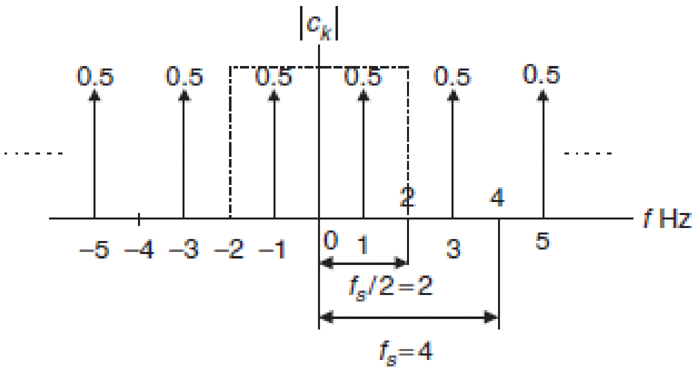


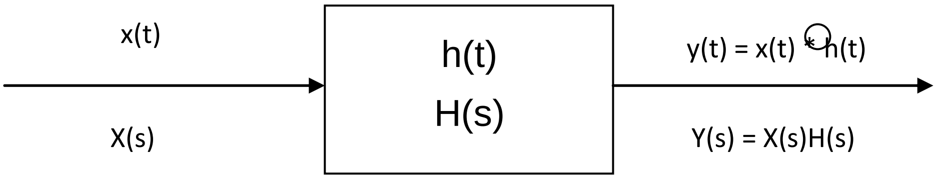
FIGURE 4.5 **Two-sided spectrum for the periodic digital signal in Example 4.1.**

As we know, the spectrum in the range of -2 to 2 Hz presents the information of the sinusoid with a frequency of 1 Hz and a peak value of 2|c1| = 1, which is converted from two sides to one side by doubling the spectral value. Note that we do not double the direct-current (DC) component, that is, c_0.

Z-Transform

8.1Introduction

A linear system can be represented in the complex frequency domain (s-domain here s = $\sigma + j\omega$) using the LaPlace Transform.



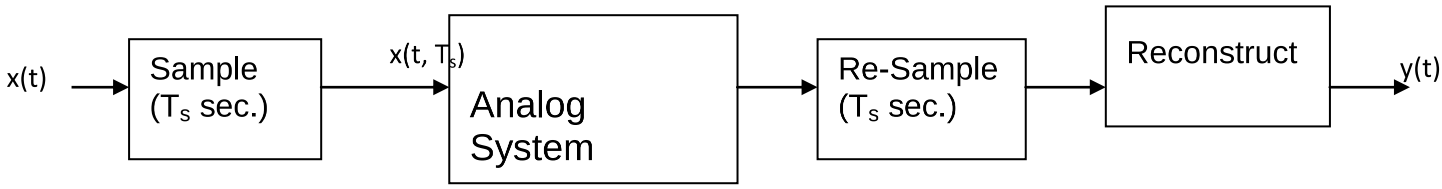
Where the direct transform is:

$$L\{x(t)\} = X(s) = \int_{t=0}^{\infty} x(t)\varepsilon^{-st}dt$$

And x(t) is assumed zero for $t \leq 0$. The Inversion integral is a contour integral in the complex plane (seldom used, tables are used instead)

$$L^{-1}\{X(s)\} = x(t) = \frac{1}{2\pi j}\int_{s=\sigma-j\infty}^{\sigma+j\infty} X(s)\varepsilon^{st}ds$$

Where σ is chosen such that the contour integral converges. If we now assume that x(t) is ideally sampled as in:



Where: $x_n = x(n*T_s) = x(t)\big|_{t=n*T_s}$ and $y_n = y(n*T_s) = y(t)\big|_{t=n*T_s}$

Analyzing this equivalent system using standard analog tools will establish the z-Transform.

4.2 Sampling

Substituting the Sampled version of x(t) into the definition of the LaPlace Transform we get

$$L\{x(t,T_s)\} = X_T(s) = \int_{t=0}^{\infty} x(t,T_s)\varepsilon^{-st}dt$$

But

$$x(t,T_s) = \sum_{n=0}^{\infty} x(t) * p(t - n*T_s)$$ (For x(t) = 0 when t < 0)

Therefore

$$X_T(s) = \int_{t=0}^{\infty} \left[\sum_{n=0}^{\infty} x(n*T_s)*\delta(t-n*T_s) \right] \varepsilon^{-st} dt$$

Now interchanging the order of integration and summation and using the sifting property of δ-functions

$$X_T(s) = \sum_{n=0}^{\infty} x(n*T_s) \int_{t=0}^{\infty} \delta(t-n*T_s) \varepsilon^{-st} dt$$

$$X_T(s) = \sum_{n=0}^{\infty} x(n*T_s) \varepsilon^{-nT_s s}$$

(We are assuming that the first sample occurs at t = 0+)

if we now adjust our nomenclature by letting:

$z = \varepsilon^{sT}$, $x(n*Ts) = x_n$, and $X(z) = X_T(s)\big|_{z=\varepsilon^{sT}}$

$$\boxed{X(z) = \sum_{n=0}^{\infty} x_n z^{-n}}$$

4.3 Which is the direct z-transform (one-sided; it assumes $x_n = 0$ for $n < 0$).

The inversion integral is:

$$\boxed{x_n = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz}$$

(This is a contour integral in the complex z-plane)

(The use of this integral can be avoided as tables can be used to invert the transform.)

To prove that these form a transform pair we can substitute one into the other.

$$x_k = \frac{1}{2\pi j} \oint_c \left[\sum_{n=0}^{\infty} x_n z^{-n} \right] z^{k-1} dz$$

Now interchanging the order of summation and integration (valid if the contour followed stays in the region of convergence):

$$x_k = \frac{1}{2\pi j} \sum_{n=0}^{\infty} x_n \oint_c z^{k-n-1} dz$$

If “C” encloses the origin (that’s where the pole is), the Cauchy Integral theorem says:

$\oint_c z^{k-n-1}dz = \begin{matrix} o & for\, n \neq k \\ 2\pi j & for\, n = k \end{matrix}$

4.4 Properties of the *z* transform

For the following

$$Z\{f[n]\} = \sum_{n=0}^{n=\infty} f[n]z^{-n} = F(z) \quad Z\{g_n\} = \sum_{n=0}^{n=\infty} g_n z^{-n} = G(z)$$

- **Linearity:**

$Z\{af_n + bg_n\} = aF(z) + bG(z)$. and ROC is $R_f \cap R_g$

which follows from definition of *z*-transform.

- **Time Shifting**

If we have $f[n] \Leftrightarrow F(z)$ then $f[n-n_0] \Leftrightarrow z^{-n_0}F(z)$

The ROC of $Y(z)$ is the same as $F(z)$ except that there are possible pole additions or deletions at $z = 0$ or $z = \infty$.

Proof:

Let $y[n] = f[n-n_0]$ then

$$Y(z) = \sum_{n=-\infty}^{\infty} f[n-n_0]z^{-n}$$

Assume $k = n - n_0$ then $n = k + n_0$, substituting in the above equation we have:

$$Y(z) = \sum_{k=-\infty}^{\infty} f[k]z^{-k-n_0} = z^{-n_0}F[z]$$

- **Multiplication by an Exponential Sequence**

Let $y[n] = z_0^n f[n]$ then $Y(z) = X\left(\frac{z}{z_0}\right)$

The consequence is pole and zero locations are scaled by z_0. If the ROC of $FX(z)$ is $r_R < |z| < r_L$, then the ROC of $Y(z)$ is

$r_R < |z/z_0| < r_L$, *i.e.*, $|z_0|r_R < |z| < |z_0|r\mathrm{L}$

Proof:

$$Y(z) = \sum_{n=-\infty}^{\infty} z_0^n x[n] z^{-n} = \sum_{n=-\infty}^{\infty} x[n]\left(\frac{z}{z_0}\right)^{-n} = X\left(\frac{z}{z_0}\right)$$

The consequence is pole and zero locations are scaled by z_0. If the ROC of $X(z)$ is $rR<|z|<rL$, then the ROC of $Y(z)$ is

$rR < |z/z_0| < rL$, *i.e.*, $|z_0|rR < |z| < |z_0|rL$

- **Differentiation of *X*(*z*)**

If we have $f[n] \Leftrightarrow F(z)$ then $nf[n] \xleftrightarrow{z} -z\dfrac{dF(z)}{z}$ and ROC = R_f

Proof:

$$F(z) = \sum_{n=-\infty}^{\infty} f[n] z^{-n}$$

$$-z\frac{dF(z)}{dz} = -z\sum_{n=-\infty}^{\infty} -n\, f[n] z^{-n-1} = \sum_{n=-\infty}^{\infty} -n\, f[n] z^{-n}$$

$$-z\frac{dF(z)}{dz} \xleftrightarrow{z} nf[n]$$

- **Conjugation of a Complex Sequence**

If we have $f[n] \Leftrightarrow F(z)$ then $f^*[n] \xleftrightarrow{z} F^*(z^*)$ and ROC = R_f

Proof:

Let $y[n] = f^*[n]$, then

$$Y(z) = \sum_{n=-\infty}^{\infty} f^*[n] z^{-n} = \left(\sum_{n=-\infty}^{\infty} f[n][z^*]^{-n}\right)^* = F^*(z^*)$$

- **Time Reversal**

If we have $f[n] \Leftrightarrow F(z)$ then $f^*[-n] \xleftrightarrow{Z} F^*\left(1/z^*\right)$

Let $y[n] = f^*[-n]$, then

$$Y(z) = \sum_{n=-\infty}^{\infty} f^*[-n] z^{-n} = \left(\sum_{n=-\infty}^{\infty} f[-n] [z^*]^{-n} \right)^* = \left(\sum_{k=-\infty}^{\infty} f[k] (1/z^*)^{-k} \right)^* = F^*(1/z^*)$$

If the ROC of $F(z)$ is $r_R < |z| < r_L$, then the ROC of $Y(z)$ is

$$r_R < \left|1/z^*\right| < r_L \; i.e., \qquad \frac{1}{r_R} > |z| > \frac{1}{r_L}$$

When the time reversal is without conjugation, it is easy to show

$$f[-n] \xleftrightarrow{Z} F(1/z) \quad \text{and ROC is } \frac{1}{r_R} > |z| > \frac{1}{r_L}$$

A comprehensive summery for the z-transform properties is shown in Table 2

Table 2 Summery of z-transform properties

Property	Sequence	z-Transform	Region of Convergence
Linearity	$ax(n) + by(n)$	$aX(z) + bY(z)$	Contains $R_x \cap R_y$
Shift	$x(n - n_0)$	$z^{-n_0} X(z)$	R_x
Time reversal	$x(-n)$	$X(z^{-1})$	$1/R_x$
Exponentiation	$\alpha^n x(n)$	$X(\alpha^{-1} z)$	$\|\alpha\| R_x$
Convolution	$x(n) * y(n)$	$X(z)Y(z)$	Contains $R_x \cap R_y$
Conjugation	$x^*(n)$	$X^*(z^*)$	R_x
Derivative	$nx(n)$	$-z \dfrac{dX(z)}{dz}$	R_x

Note: Given the z-transforms $X(z)$ and $Y(z)$ of $x(n)$ and $y(n)$, with regions of convergence R_x and R_y, respectively, this table lists the z-transforms of sequences that are formed from $x(n)$ and $y(n)$.

Example 3: Find the z transform of $3n + 2 \times 3^n$.

SolutionFrom the linearity property

$Z\{3n + 2 \times 3^n\} = 3Z\{n\} + 2Z\{3^n\}$

and from the Table 1

$$Z\{n\} = \frac{z}{(z-1)^2} \quad \text{and} \quad Z\{3^n\} = \frac{z}{(z-3)}$$

($r^n$with $r = 3$). Therefore

$$Z\{3n + 2 \times 3^n\} = \frac{3z}{(z-1)^2} + \frac{2z}{(z-3)}$$

Example 4: Find the z-transform of each of the following sequences:

(a) $\boldsymbol{x(n) = 2^n u(n) + 3(½)^n u(n)}$
(b) $\boldsymbol{x(n) = \cos(n\omega_0) u(n)}$.

Solution:

(a) Because $x(n)$ is a sum of two sequences of the form $\alpha^n u(n)$, using the linearity property of the z-transform, and referring to Table 1, the z-transform pair

$$X(z) = \frac{1}{1-2z^{-1}} + \frac{3}{1-\frac{1}{2}z^{-1}} = \frac{4-\frac{13}{2}z^{-1}}{(1-2z)\left(1-\frac{1}{2}z^{-1}\right)}$$

(b) For this sequence we write
$x(n) = \cos(n\omega_0)\, u(n) = ½(e^{jn\omega 0} + e^{-jn\omega 0})\, u(n)$

Therefore, the z-transform is

$$X(z) = \frac{1}{2}\frac{1}{1-e^{jn\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-jn\omega_0}z^{-1}}$$

with a region of convergence |z| >1. Combining the two terms together, we have

$$X(z)=\frac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$$

4.5 The Inverse *z*-Transform

The *z*-transform is a useful tool in linear systems analysis. However, just as important as techniques for finding the z-transform of a sequence are methods that may be used to invert the z-transform and recover the sequence ***x*(*n*)**from ***X*(*z*)**. Three possible approaches are described below.

- **Partial Fraction Expansion**

For *z*-transforms that are rational functions of *z*,

$$X(z)=\frac{\sum_{k=0}^{q} b(k)z^{-k}}{\sum_{k=0}^{p} a(k)z^{-k}}=C\,\frac{\prod_{k=1}^{q}(1-\beta_k z^{-1})}{\prod_{k=1}^{p}(1-\alpha_k z^{-1})}$$

a simple and straightforward approach to find the inverse *z*-transform is to perform a partial fraction expansion of $X(z)$. Assuming that $p > q$, and that all of the roots in the denominator are simple, $\alpha_i \neq \alpha_k$ for $i \neq k$, $X(z)$ may be expanded as follows:

$$X(z)=\sum_{k=1}^{p}\frac{A_k}{1-\alpha_k z^{-1}} \qquad \text{Eq(3)}$$

for some constants A_k for $k = 1,2, \ldots, p$. The coefficients A_k may be found by multiplying both sides of Eq. (3) by $(1 - \alpha_k z^{-1})$ and setting $z = \alpha_k$. The result is

$$A_k=\left[(1-\alpha_k z^{-1})X(z)\right]_{z=\alpha_k}$$

If $p \leq q$, the partial fraction expansion must include a polynomial in $z^{-1}$of order $(p-q)$. The coefficients of this polynomial may be found by long division (i.e., by dividing the numerator polynomial by the denominator). For multiple-order poles, the expansion must be modified. For example, if $X(z)$ has a second-order pole at $z = \alpha_k$, the expansion will include two terms,

$$\frac{B_1}{1-\alpha_k z^{-1}} + \frac{B_2}{(1-\alpha_k z^{-1})^2}$$

where B_1,and $B_2$are given by

$$B_1 = \alpha_k \left[\frac{d}{dz}(1-\alpha_k z^{-1})^2 X(z) \right]_{z=\alpha_k}$$

$$B_2 = \left[(1-\alpha_k z^{-1})^2 X(z)\right]_{z=\alpha_k}$$

Example 5: Suppose that a sequence $x(n)$has a z-transform

$$X(z) = \frac{4 - \frac{7}{4}z^{-1} + \frac{1}{4}z^{-2}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{4 - \frac{7}{4}z^{-1} + \frac{1}{4}z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

Solution:

With a region of convergence $|z|> ½$. Because $p = q = 2$, and the two poles are simple, the partial fraction expansion has the form

$$X(z) = C + \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{1-\frac{1}{4}z^{-1}}$$

The constant C is found by long division:

$$\begin{array}{r|l}
 & \quad\quad 2 \\ \cline{2-2}
\frac{1}{8}z^{-2} - \frac{3}{4}z^{-1} + 1 & \frac{1}{4}z^{-2} - \frac{7}{4}z^{-1} + 4 \\
 & \frac{1}{4}z^{-2} - \frac{3}{2}z^{-1} + 2 \\ \cline{2-2}
 & \quad\quad -\frac{1}{4}z^{-1} + 2
\end{array}$$

Therefore, $C = 2$ and we may write $X(z)$ as follows:

$$X(z) = 2 + \frac{2 - \frac{1}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{4}z^{-1}\right)}$$

Next, for the coefficients $A_1$and A_2we have

$$A_1 = \left[\left(1-\frac{1}{2}z^{-1}\right)X(z)\right]_{z^{-1}=2} = \left.\frac{4 - \frac{7}{4}z^{-1} + \frac{1}{4}z^{-2}}{1-\frac{1}{4}z^{-1}}\right|_{z^{-1}=2} = 3$$

and

$$A_2 = \left[\left(1 - \tfrac{1}{4}z^{-1}\right)X(z)\right]_{z^{-1}=4} = \left.\frac{4 - \frac{7}{4}z^{-1} + \frac{1}{4}z^{-2}}{1 - \frac{1}{2}z^{-1}}\right|_{z^{-1}=4} = -1$$

Thus, the complete partial fraction expansion becomes

$$X(z) = 2 + \frac{3}{1 - \frac{1}{2}z^{-1}} - \frac{1}{1 - \frac{1}{4}z^{-1}}$$

Finally, because the region of convergence is the exterior of the circle |z| > 1, $x(n)$ is the right-sided sequence

$$x(n) = 2\delta(n) + 3\left(\tfrac{1}{2}\right)^n u(n) - \left(\tfrac{1}{4}\right)^n u(n)$$

- **Power Series**

The z-transform is a power series expansion,

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \cdots + x(-2)\,z^2 + x(-1)\,z + x(0) + x(1)\,z^{-1} + x(2)\,z^{-2} + \cdots$$

where the sequence values $x(n)$are the coefficients of z^{-n} in the expansion. Therefore, if we can find the power series expansion for $X(z)$, the sequence values $x(n)$may be found by simply picking off the coefficients of z^{-n}.

Example 6: Consider the z-transform

$$X(z) = \log(1 + az^{-1}) \qquad |z| > |a|$$

Solution:

The power series expansion of this function is

$$\log(1 + az^{-1}) = \sum_{n=1}^{\infty} \frac{1}{n}(-1)^{n+1} a^n z^{-n}$$

Therefore, the sequence x(n) having this z-transform is

$$x(n) = \begin{cases} \frac{1}{n}(-1)^{n+1}a^n & n > 0 \\ 0 & n \le 0 \end{cases}$$

- **Contour Integration**

Another approach that may be used to find the inverse z-transform of $X(z)$ is to use contour integration. This procedure relies on Cauchy's integral theorem, which states that if C is a closed contour that encircles the origin in a counterclockwise direction,

$$\frac{1}{2\pi j}\oint_C z^{-k} dz = \begin{cases} 1 & k = 1 \\ 0 & k \neq 1 \end{cases}$$

With

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Cauchy's integral theorem may be used to show that the coefficients $x(n)$ may be found from $X(z)$ as follows:

$$x(n) = \frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz$$

where $C$is a closed contour within the region of convergence of $X(z)$ that encircles the origin in a counterclockwise direction. Contour integrals of this form may often by evaluated with the help of Cauchy's residue theorem,

$$x(n) = \frac{1}{2\pi j}\oint_C X(z) z^{n-1}\, dz = \sum \left[\text{residues of } X(z) z^{n-1} \text{at the poles inside } C\right]$$

If $X(z)$ is a rational function of z with a first-order pole at $z = \alpha_k$,

$$\text{Res}\left[X(z) z^{n-1} \text{ at } z = \alpha_k\right] = \left[(1 - \alpha_k z^{-1}) X(z) z^{n-1}\right]_{z=\alpha_k}$$

Contour integration is particularly useful if only a few values of *x*(*n*) are needed.

Example 7:

Find the inverse of each of the following *z*-transforms:

$$(a)\ \ X(z) = 4 + 3(z^2 + z^{-2}) \qquad 0 < |z| < \infty$$

$$(b)\ \ X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{3}{1 - \frac{1}{3}z^{-1}} \qquad |z| > \tfrac{1}{2}$$

$$(c)\ \ X(z) = \frac{1}{1 + 3z^{-1} + 2z^{-2}} \qquad |z| > 2$$

$$(d)\ \ X(z) = \frac{1}{(1 - z^{-1})(1 - z^{-2})} \qquad |z| > 1$$

Solution:

a) Because $X(z)$ is a finite-order polynomial, $x(n)$ is a finite-length sequence. Therefore, $x(n)$ is the coefficient that multiplies z^{-1} in $X(z)$. Thus, $x(0) = 4$ and $x(2) = x(-2) = 3$.

b) This *z*-transform is a sum of two first-order rational functions of *z*. Because the region of convergence of $X(z)$ is the exterior of a circle, $x(n)$ is a right-sided sequence. Using the *z*-transform pair for a right-sided exponential, we may invert $X(z)$ easily as follows:

$$x(n) = \left(\tfrac{1}{2}\right)^n u(n) + 3\left(\tfrac{1}{3}\right)^n u(n)$$

c) Here we have a rational function of *z* with a denominator that is a quadratic in *z*. Before we can find the inverse z-transform, we need to factor the denominator and perform a partial fraction expansion:

$$X(z) = \frac{1}{1 + 3z^{-1} + 2z^{-2}} = \frac{1}{(1 + 2z^{-1})(1 + z^{-1})}$$

$$= \frac{2}{1 + 2z^{-1}} - \frac{1}{1 + z^{-1}}$$

Because $x(n)$ is right-sided, the inverse *z*-transform is

$$x(n) = 2(-2)^n u(n) - (-1)^n u(n)$$

d) One way to invert this *z*-transform is to perform a partial fraction expansion. With

$$X(z) = \frac{1}{(1-z^{-1})(1-z^{-2})} = \frac{1}{(1-z^{-1})^2(1+z^{-1})}$$

$$= \frac{A}{1+z^{-1}} + \frac{B_1}{1-z^{-1}} + \frac{B_2}{(1-z^{-1})^2}$$

the constants A, B_1, and $B_2$are as follows:

$$A = [(1+z^{-1})X(z)]_{z=-1} = \tfrac{1}{4}$$

$$B_1 = \left[\frac{d}{dz}(1-z^{-1})^2 X(z)\right]_{z=1} = \left[\frac{z^{-2}}{(1+z^{-1})^2}\right]_{z=1} = \tfrac{1}{4}$$

$$B_2 = [(1-z^{-1})^2 X(z)]_{z=1} = \tfrac{1}{2}$$

Inverse transforming each term, we have

$$x(n) = \tfrac{1}{4}[(-1)^n + 1 + 2(n+1)]u(n)$$

Example 7:

Find the inverse z-transform of the second-order system

$$X(z) = \frac{1+\frac{1}{4}z^{-1}}{\left(1-\frac{1}{2}z^{-1}\right)^2} \qquad |z| > \tfrac{1}{2}$$

Here we have a second-order pole at z = ½. The partial fraction expansion for $X(z)$ is

$$X(z) = \frac{A_1}{1-\frac{1}{2}z^{-1}} + \frac{A_2}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

The constant A_1 is

$$A_1 = \tfrac{1}{2}\left[\frac{d}{dz}\left(1-\tfrac{1}{2}z^{-1}\right)^2 X(z)\right]_{z=1/2} = \tfrac{1}{2}\left[-\tfrac{1}{4}z^{-2}\right]_{z=1/2} = -\tfrac{1}{2}$$

and the constant A_2 is

$$A_2 = \left[\left(1-\tfrac{1}{2}z^{-1}\right)^2 X(z)\right]_{z=1/2} = \tfrac{3}{2}$$

Therefore,

$$X(z) = -\frac{\frac{1}{2}}{1-\frac{1}{2}z^{-1}} + \frac{\frac{3}{2}}{\left(1-\frac{1}{2}z^{-1}\right)^2}$$

and

$$x(n) = -\left(\tfrac{1}{2}\right)^{n+1}u(n) + 3(n+1)\left(\tfrac{1}{2}\right)^{n+1}u(n)$$

Example 8:

Find the inverse z-transform of $X(z) = \sin z$.

Solution

To find the inverse z-transform of $X(z) = \sin z$, we expand $X(z)$ in a Taylor series about $z = 0$ as follows:

$$\begin{aligned} X(z) &= X(z)\Big|_{z=0} + z\frac{dX(z)}{dz}\Big|_{z=0} + \frac{z^2}{2!}\frac{d^2X(z)}{dz^2}\Big|_{z=0} + \cdots + \frac{z^n}{n!}\frac{d^nX(z)}{dz^n}\Big|_{z=0} + \cdots \\ &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{z^{2n+1}}{(2n+1)!} \end{aligned}$$

Because

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

we may associate the coefficients in the Taylor series expansion with the sequence values $x(n)$. Thus, we have

$$x(n) = (-1)^n\frac{1}{(2|n|+1)!} \qquad n = -1, -3, -5, \ldots$$

Example 8:

Evaluate the following integral:

$$\frac{1}{2\pi j}\oint_C \frac{1+2z^{-1}-z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)} z^3 dz$$

where the contour of integration *C* is the unit circle.

Solution:

Recall that for a sequence *x*(*n*) that has a *z*-transform *X*(*z*), the sequence may be recovered using contour integration as follows:

$$x(n) = \frac{1}{2\pi j}\oint_c X(z) z^{n-1} dz$$

Therefore, the integral that is to be evaluated corresponds to the value of the sequence *x*(*n*) at *n* = 4 that has a *z*-transform

$$X(z) = \frac{1+2z^{-1}-z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{2}{3}z^{-1}\right)}$$

Thus, we may find *x*(*n*) using a partial fraction expansion of *X*(*z*) and then evaluate the sequence at *n* = 4. With this approach, however, we are finding the values of *x*(*n*) for all n. Alternatively, we could perform long division and divide the numerator of *X*(*z*) by the denominator. The coefficient multiplying z^{-4} would then be the value of *x*(*n*) at *n* = 4, and the value of the integral. However, because we are only interested in the value of the sequence at *n* = 4, the easiest approach is to evaluate the integral directly using the Cauchy integral theorem. The value of the integral is equal to the sum of the residues of the poles of $X(z)z^3$ inside the unit circle. Because

$$X(z)z^3 = z^3 \frac{z^2+2z-1}{\left(z-\frac{1}{2}\right)\left(z-\frac{2}{3}\right)}$$

has poles at z =1/2 and z =2/3,

$$\text{Res}[X(z)z^3]_{z=\frac{1}{2}} = \left[z^3 \frac{z^2+2z-1}{z-\frac{2}{3}}\right]_{z=\frac{1}{2}} = -\frac{3}{16}$$

and

$$\text{Res}[X(z)z^3]_{z=\frac{2}{3}} = \left[z^3 \frac{z^2+2z-1}{z-\frac{1}{2}}\right]_{z=\frac{2}{3}} = \tfrac{112}{81}$$

Therefore, we have

$$\frac{1}{2\pi j}\oint_c X(z)z^3 dz = \tfrac{112}{81} - \tfrac{3}{16} = 1.1952$$

PROPERTIES OF DISCRETE FOURIER TRANSFORM

As a special case of general Fourier transform, the discrete time transform shares all properties (and their proofs) of the Fourier transform discussed above, except now some of these properties may take different forms. In the following, we always assume $\mathcal{F}[x[m]] = X(e^{j\omega})$ and $\mathcal{F}[y[m]] = Y(e^{j\omega})$.

- **Linearity**

$$\mathcal{F}[ax[m]+by[m]] = aX(e^{j\omega}) + bY(e^{j\omega})$$

- **Time Shifting**

$$\mathcal{F}[x[m-m_0]] = e^{-jm_0\omega}X(e^{j\omega})$$

Proof:

$$\mathcal{F}[x[m-m_0]] = \sum_{m=-\infty}^{\infty} x[m-m_0]e^{-j\omega m}$$

If we let $m' = m - m_0$, the above becomes

$$\mathcal{F}[x[m - m_0]] = \sum_{m=-\infty}^{\infty} x[m'] e^{-j\omega(m' + m_0)} = e^{-j\omega m_0} X(e^{j\omega})$$

- **Time Reversal**

$$\mathcal{F}[x[-m]] = X(e^{-j\omega})$$

- **Frequency Shifting**

$$\mathcal{F}[x[m] e^{j\omega_0 m}] = X(e^{j(\omega - \omega_0)})$$

- **Differencing**

Differencing is the discrete-time counterpart of differentiation.

$$\mathcal{F}[x[m] - x[m-1]] = (1 - e^{-j\omega}) X(e^{j\omega})$$

Proof:

$\mathcal{F}[x[m] - x[m-1]] = \mathcal{F}[x[m]] - \mathcal{F}[x[m-1]]$

$X(e^{j\omega}) - X(e^{j\omega}) e^{-j\omega} = (1 - e^{-j\omega}) X(e^{j\omega})$

- **Differentiation in frequency**

$$\mathcal{F}^{-1}[j\frac{d}{d\omega}X(e^{j\omega})] = m\ x[m]$$

proof: Differentiating the definition of discrete Fourier transform with respect to ω, we get

$$\frac{d}{d\omega}X(e^{j\omega}) \quad : \quad \frac{d}{d\omega}\sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m} = \sum_{m=-\infty}^{\infty} x[m]\frac{d}{d\omega}e^{-j\omega m}$$

$$: \quad \sum_{m=-\infty}^{\infty} -jmx[m]e^{-j\omega m}$$

- **Convolution Theorems**

The convolution theorem states that convolution in time domain corresponds to multiplication in frequency domain and vice versa:

$$\mathcal{F}[x[n] * y[n]] = X(e^{j\omega})\ Y(e^{j\omega}) \qquad (a)$$

$$\mathcal{F}[x[n]\ y[n]] = X(e^{j\omega}) * Y(e^{j\omega}) \qquad (b)$$

Recall that the convolution of periodic signals $x_T(t)$ and $y_T(t)$ is

$$x_T(t) * y_T(t) \triangleq \frac{1}{T}\int_T x_T(\tau)y_T(t-\tau)d\tau$$

Here the convolution of periodic spectra $X(f)$ and $Y(f)$ is similarly defined as

$$X(e^{j\omega})*Y(e^{j\omega}) = \frac{1}{\Omega}\int_{\Omega} X(e^{j\omega'})Y(e^{j(\omega-\omega')})\,d\omega' = \frac{1}{2\pi}\int_{0}^{2\pi} X(e^{j\omega'})Y(e^{j(\omega-\omega')})\,d\omega'$$

Proof of (a):

$$\mathcal{F}[x[n]*y[n]] \quad : \quad \sum_{n=-\infty}^{\infty}[\sum_{m=-\infty}^{\infty} x[m]y[n-m]\,]e^{-jn\omega}$$

$$: \quad \sum_{m=-\infty}^{\infty} x[m][\sum_{n=-\infty}^{\infty} y[n-m]e^{-j(n-m)\omega}]e^{-jm\omega}$$

$$: \quad X(j\omega)\;Y(j\omega)$$

Proof of (b):

$$\mathcal{F}[x[n]y[n]] \; : \; \sum_{n=-\infty}^{\infty} x[n]y[n]e^{-jn\omega} = \sum_{n=-\infty}^{\infty}[\frac{1}{2\pi}\int_0^{2\pi} X(j\omega')e^{jn\omega'}d\omega']y[n]e^{-jn\omega}$$

$$: \quad \frac{1}{2\pi}\int_0^{2\pi} X(j\omega')[\sum_{n=-\infty}^{\infty} e^{jn\omega'}y[n]e^{-jn\omega}]d\omega'$$

$$: \quad \frac{1}{2\pi}\int_0^{2\pi} X(j\omega')\sum_{n=-\infty}^{\infty} y[n]e^{-jn(\omega-\omega')}d\omega'$$

$$: \quad \frac{1}{2\pi}\int_0^{2\pi} X(j\omega')Y(j(\omega-\omega'))d\omega' = X(j\omega)*Y(j\omega)$$

- **Parseval's Relation**

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_0^{2\pi} |X(e^{j\omega})|^2 d\omega$$

The **circular convolution**, also known as **cyclic convolution**, of two aperiodic functions occurs when one of them is convolved in the normal way with a periodic summation of the other function. That situation arises in the context of the Circular convolution theorem. The identical operation can also be expressed in terms of the periodic summations of both functions, if the infinite integration interval is reduced to just one period. That situation arises in the context of the discrete-time Fourier transform (DTFT) and is also called **periodic convolution**. In particular, the transform (DTFT) of the product of two discrete sequences is the periodic convolution of the transforms of the individual sequences.

For a periodic function x_T, with period T, the convolution with another function, h, is also periodic, and can be expressed in terms of integration over a finite interval as follows**:**

For a periodic function x_T, with period T, the convolution with another function, h, is also periodic, and can be expressed in terms of integration over a finite interval as follows**:**

$$\begin{aligned}(x_T * h)(t) \quad &\stackrel{\text{def}}{=} \int_{-\infty}^{\infty} h(\tau)\cdot x_T(t-\tau)\,d\tau \\ &= \int_{t_o}^{t_o+T} h_T(\tau)\cdot x_T(t-\tau)\,d\tau,\end{aligned}$$ [2]

where t_o is an arbitrary parameter, and h_T is a periodic summation of h, defined by**:**

$$h_T(t) \stackrel{\text{def}}{=} \sum_{k=-\infty}^{\infty} h(t-kT) = \sum_{k=-\infty}^{\infty} h(t+kT).$$

This operation is a **periodic convolution** of functions x_T and h_T. When x_T is expressed as the periodic summation of another function, x, the same operation may also be referred to as a **circular convolution** of functions h and x.

Discrete sequences

Similarly, for discrete sequences and period **N**, we can write the **circular convolution** of functions *h* and *x* as**:**

$$(x_N * h)[n] \stackrel{\text{def}}{=} \sum_{m=-\infty}^{\infty} h[m] \cdot x_N[n-m]$$
$$= \sum_{m=-\infty}^{\infty} \left(h[m] \cdot \sum_{k=-\infty}^{\infty} x[n-m-kN] \right).$$

This corresponds to matrix multiplication, and the kernel of the integral transform is a circular matrix

 ^ If a sequence, *x*[*n*], represents samples of a continuous function, *x*(*t*), with Fourier transform *X*(*f*), its DTFT is a periodic summation of *X*(*f*).

 ^ Proof**:**

$$\int_{-\infty}^{\infty} h(\tau) \cdot x_T(t-\tau)\, d\tau$$
$$= \sum_{k=-\infty}^{\infty} \left[\int_{t_o+kT}^{t_o+(k+1)T} h(\tau) \cdot x_T(t-\tau)\ d\tau \right]$$
$$\stackrel{\tau \rightarrow \tau+kT}{=} \sum_{k=-\infty}^{\infty} \left[\int_{t_o}^{t_o+T} h(\tau+kT) \cdot x_T(t-\tau-kT)\ d\tau \right]$$
$$= \int_{t_o}^{t_o+T} \left[\sum_{k=-\infty}^{\infty} h(\tau+kT) \cdot \underbrace{x_T(t-\tau-kT)}_{X_T(t-\tau),\text{ by periodicity}} \right] d\tau$$
$$= \int_{t_o}^{t_o+T} \underbrace{\left[\sum_{k=-\infty}^{\infty} h(\tau+kT) \right]}_{\stackrel{\text{def}}{=}\ h_T(\tau)} \cdot x_T(t-\tau)\ d\tau \qquad (QED)$$

Definition of the Fourier Transform
The Fourier transform (FT) of the function *f .(x)* is the function *F(ω)* where:

$$F(\omega) = \int_{-\infty}^{\infty} f(x) e^{-i\omega x}\, dx$$

and the inverse Fourier transform is

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega x}\, d\omega$$

Recall that $i = \sqrt{-1}$ and $e^{i\theta} = \cos\theta + i \sin\theta$.

Think of it as a transformation into a different set of basis functions. The Fourier transform uses complex exponentials (sinusoids) of various frequencies as its basis functions.(Other transforms, such as Z, Laplace, Cosine, Wavelet, and Hartley, use different basic functions).
A Fourier transform A Fourier transform pair is often written $f(x) \leftrightarrow F(\omega)$, or $F(f(x)) = F(\omega)$ where *F* is the Fourier transform operator. If *f .x/* is thought of as a signal (i.e. input data) then we call *F(ω)*the signal's *spectrum*. If *f* is thought of as the *impulse response* of a filter (which operates on input data to produce output data) then we call *F* the filter's *frequency response*. (Occasionally the line between what's signal and what's filter becomes blurry).

Example of a Fourier Transform

Suppose we want to create a filter that eliminates high frequencies but retains low frequencies (this is very useful in anti aliasing). In signal processing terminology, this is called an *ideal low pass filter*. So we'll specify a box-shaped frequency response with cutoff frequency ω_C

$$F(\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

What is its impulse response?

We know that the impulse response is the inverse Fourier transform of the frequency response, so taking off our signal processing hat and putting on our mathematics hat, all we need to do is evaluate:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega x}\, d\omega$$

for this particular $F(\omega)$:

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{i\omega x}\, d\omega \\ &= \frac{1}{2\pi}\frac{e^{i\omega x}}{ix}\Big|_{\omega=-\omega_c}^{\omega_c} \\ &= \frac{1}{\pi x}\frac{e^{i\omega_c x} - e^{-i\omega_c x}}{2i} \\ &= \frac{\sin \omega_c x}{\pi x} \qquad \text{since } \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \\ &= \frac{\omega_c}{\pi}\mathrm{sinc}\left(\frac{\omega_c}{\pi}x\right) \end{aligned}$$

where $\mathrm{sinc}(x) = \sin(\pi x)/(\pi x)$. For antialiasing with unit-spaced samples, you want the cutoff frequency to equal the Nyquist frequency, so $\omega_c = \pi$.

Fourier Transform Properties

Rather than write "the Fourier transform of an X function is a Y function", we write the shorthand: $X \leftrightarrow Y$. If z is a complex number and $z = x + iy$ where x and y are its real and imaginary parts, then the complex conjugate of z is $z^* = x - iy$. A function $f(u)$ is *even* if $f(u) = f(-u)$, it is *odd* if $f(u) = -f(-u)$, it is conjugate symmetric if $f(u) = f^*(-u)$, and it is conjugate antisymmetric if $f(u) = -f^*(-u)$.

Convolution Theorem

The Fourier transform of a convolution of two signals is the product of their Fourier transforms: $f \circledast g \leftrightarrow FG.$. The convolution of two continuous signals f and g is

$$(f \circledast g)(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)\,dt$$

So $\int_{-\infty}^{+\infty} f(t)g(x-t)\,dt \leftrightarrow F(\omega)G(\omega)$.

The Fourier transform of a product of two signals is the convolution of their Fourier transforms: $fg \leftrightarrow F \circledast G/2\pi$.

Delta Functions

The (Dirac) delta function $\delta(x)$ is defined such that $\delta(x) = 0$ for all $x \neq 0$, $\int_{-\infty}^{+\infty} \delta(t)\,dt = 1$, and for any $f(x)$:

$$(f \circledast \delta)(x) = \int_{-\infty}^{+\infty} f(t)\delta(x-t)\,dt = f(x)$$

The latter is called the *sifting property* of delta functions. Because convolution with a delta is linear shift-invariant filtering, translating the delta by a will translate the output by a:

$$\Big(f(x) \circledast \delta(x-a)\Big)(x) = f(x-a)$$

DISCRETE FOURIER TRANSFORM

In time domain, representation of digital signals describes the signal amplitude versus the sampling time instant or the sample number. However, in some applications, signal frequency content is very useful otherwise than as digital signal samples. The representation of the digital signal in terms of its frequency component in a frequency domain, that is, the signal spectrum, needs to be developed. As an example, Figure 4.1 illustrates the time domain representation
of a 1,000-Hz sinusoid with 32 samples at a sampling rate of 8,000 Hz; the bottom plot shows the signal spectrum (frequency domain representation), where we can clearly observe that the amplitude peak is located at the frequency of 1,000 Hz in the calculated spectrum. Hence, the spectral plot better displays frequency information of a digital signal.

The algorithm transforming the time domain signal samples to the frequency domain components is known as the discrete Fourier transform, or DFT. The DFT also establishes a relationship between the time domain representation and the frequency domain representation. Therefore, we can apply the DFT to perform frequency analysis of a time domain sequence. In addition, the DFT is widely used in many other areas, including spectral analysis, acoustics, imaging/video, audio, instrumentation, and communications systems. To be able to develop the DFT and understand how to use it, we first study the spectrum of periodic digital signals using the Fourier series.

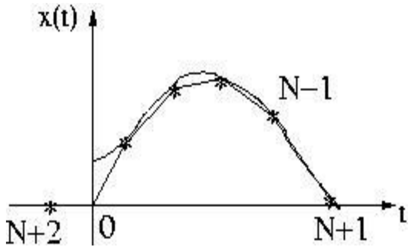
Consider a finite duration signal $g(t)$ of duration T sampled at a uniform rate t_s such that

$T = Nt_s$ where N is an integer $N > 0$

Then the Fourier transform of signal is given by

$$G(f) = \int_0^T g(t)e^{-j2\pi ft}\, dt$$

If we now evaluate the above integral by trapezoidal rule of integration after padding two zeros at the extremity on either side [signal is zero there infact, we obtain the following expressions.



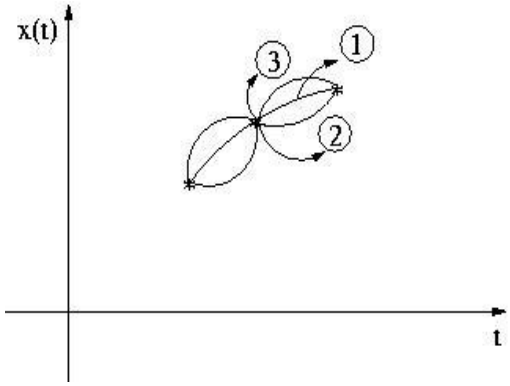
$$G(f) = t_s \sum_{n=0}^{N-1} g(nt_s)e^{-j2\pi fnt_s} \qquad \longrightarrow (1)$$

The inverse DFT (IDFT) which is used to reconstruct the signal is given by:

$$g(t) = \int_{-\infty}^{\infty} G(f)\, e^{j2\pi ft}\, df \longrightarrow (2)$$

Fig (1)

If, from equation (1) we could compute complete frequency spectrum i.e. $G(f), -\infty \le f \le \infty$ then (2) would imply that we can obtain $g(t) \forall 0 \le t \le T$. The fallacy in the above statement is quite obvious as we have only finite samples and the curve connecting any 2-samples can be defined plausibly in infinitely many ways (see fig (2)). This suggests that from (1), we should be able to derive only limited amount of frequency domain information. Since, we have N-data points [real] and $G(f)$ a complex number contains both magnitude and phase angle information in the frequency domain (2-units of information), it is reasonable to expect that we should be in a position to redict atmost $\frac{N}{2}$ transforms $G(f)$



for original signal.

Now, let $f_0 = \frac{1}{T} = \frac{1}{Nt_s} = \frac{f_s}{N}$

and $f = mf_0 = \frac{m}{Nt_s} = \frac{mf_s}{N} \longrightarrow$ (3)

then substituting (3) in (1), we get

$$G\left(\frac{mf_s}{N}\right) = t_s \sum_{n=0}^{N-1} g(nt_s)\, e^{-j2\pi \frac{m}{Nt_s} nt_s}$$

$$= t_s \sum_{n=0}^{N-1} g(nt_s)\, e^{\frac{-j2\pi mn}{N}}$$

Note that our choice of frequency is such that the exponential term in (1) is independent of t_s. The intuition for choosing such f is that, basically we are attempting a transform on discrete samples which may (or) may not have a corresponding analog 'parent' signal. This suggests to us the following discrete version of Fourier transform for a discrete sequence $\{x_0, x_1, \ldots\ldots, x_{N-1}\}$

$$X(m) = \sum_{n=0}^{N-1} x(n) e^{\frac{-j2\pi mn}{N}} \longrightarrow (4)$$

Our next job should be to come up with inverse transformation. Assuming for N-samples $x_0, x_1, \ldots\ldots\ldots x_{N-1}$ that (4) would be a transformation and if (2) defines IFT in continuous domain, in the discrete domain, we can hypothesize following inverse transform.

$$x(n) = \frac{1}{K} \sum_{m=0}^{N-1} X(m) e^{\frac{j2\pi mn}{N}} \longrightarrow (5)$$

Where K is a suitable scaling factor.

Our next job is to verify that (4) and (5) indeed define a transformation pair

Substituting (4) in (5), we get following expression for right hand side of (5)

$$\text{Right hand side} = \frac{1}{K} \sum_{m=0}^{N-1} \left[\sum_{k=0}^{N-1} x(k) e^{\frac{-j2\pi mk}{N}} \right] e^{\frac{j2\pi mn}{N}} \longrightarrow (6)$$

[Note the use of dummy subscript k]

Let us work this expression out in a long hand fashion; for compactness we use notation $x_n = x(n)$

$$RHS = \frac{1}{K} \left[\begin{array}{lr} x_0 + x_1 + ------ + x_{N-1} & \leftarrow \quad m = 0 \\ + x_0 e^{\frac{j2\pi n}{N}} + x_1 e^{\frac{-j2\pi}{N}} e^{\frac{j2\pi n}{N}} + --- + x_{N-1} e^{\frac{-j2\pi(N-1)}{N}} e^{\frac{j2\pi n}{N}} & \leftarrow \quad m = 1 \\ + ---------------------------- + & \\ + x_0 e^{\frac{j2\pi(N-1)n}{N}} + x_1 e^{\frac{-j2\pi(N-1)}{N}} e^{\frac{j2\pi(N-1)n}{N}} + ---- + x_{N-1} e^{\frac{-j2\pi(N-1)^2}{N}} e^{\frac{j2\pi(N-1)n}{N}} & \leftarrow \quad m = N-1 \end{array} \right]$$

In the above expression, for the first row m is set to zero, for the second row it is set to one and for the last row $m = N-1$

Now, grouping terms column wise, we get

$$RHS = \frac{1}{K}\left[\begin{array}{l} x_0(1+e^{\frac{j2\pi n}{N}} + - - - - + e^{\frac{j2\pi(N-1)n}{N}}) + x_1(1+e^{\frac{j2\pi(n-1)}{N}} + - - - - + e^{\frac{j2\pi(N-1)(n-1)}{N}}) \\ + - - - - - - + x_{N-1}(1+e^{\frac{j2\pi}{N}(n-\overline{N-1})} + - - - - e^{\frac{j2\pi(N-1)}{N}(n-\overline{N-1})}) \end{array}\right]$$

$$= \frac{1}{K}\sum_{k=0}^{N-1} x_k \sum_{m=0}^{N-1} e^{\frac{j2\pi m}{N}(n-k)}$$

Note that this jugglery shows that we can interchange the summation order. One order indicates row wise and another column wise summation

i.e. $$RHS = \frac{1}{K}\sum_{k=0}^{N-1}\sum_{m=0}^{N-1} x_k e^{\frac{j2\pi m}{N}(n-k)} \longrightarrow \quad (7)$$

Our primary task now is to evaluate the expression.

$$\sum_{m=0}^{N-1} e^{\frac{j2\pi m}{N}(n-k)}$$

We now claim that

$$\sum_{m=0}^{N-1} e^{\frac{j2\pi}{N}(n-k)m} = \left.\begin{array}{l} = N \\ = 0 \end{array}\right\} \begin{array}{l} if\ \ k = n; \\ if\ \ k \neq n; \end{array}$$

Proof: - for $k = n$ $e^{\frac{j2\pi}{N}(n-k)m} = e^{j.0.m} = 1 \ \forall m$

Hence, the first case is obvious.

Now, if $k \neq n$, let $k - n = k^1$

$$\sum_{m=0}^{N-1} e^{\frac{j2\pi m}{N}(n-k)} = e^{\frac{j2\pi}{N}(n-k)0} + e^{\frac{j2\pi}{N}(n-k)1} + - - - - + e^{\frac{j2\pi}{N}(n-k)(N-1)}$$

Now, $e^{\frac{j2\pi}{N}k^1 m} = \left(e^{\frac{j2\pi}{N}k^1}\right)^m = \left(a^{k^1}\right)^m$

where $a = e^{\frac{j2\pi}{N}}$

$$\therefore \quad \sum_{m=0}^{N-1} e^{\frac{j2\pi m}{N}(n-k)} = \sum_{m=0}^{N-1}\left(e^{j\frac{2\pi}{N}k^1}\right)^m = \frac{1-e^{\left(\frac{j2\pi}{N}k^1\right)N}}{1-e^{\frac{j2\pi}{N}k^1}} = \frac{1-e^{j2\pi k^1}}{1-e^{\frac{j2\pi}{N}k^1}} = 0$$

$\because$ if k^1 is integer, then $e^{j2\pi k^1} = 1$

Note that we have used the following geometric series expression $a + ar + \ldots\ldots + ar^{n-1} = \dfrac{a(1-r)^n}{1-r}$

Thus, RHS in (6) is equal to $\dfrac{N}{K}x_n$

We see that equation (6) defines the inverse transformation if we choose $K = N$;

Thus, N-point DFT and IDFT for samples $[x_0, x_1, \ldots\ldots x_{N-1}]$ are defined as follows.

$$X(m) = \sum_{n=0}^{N-1} x_n e^{\frac{-j2\pi nm}{N}}$$

$$x(n) \quad = \frac{1}{N}\sum_{m=0}^{N-1} X(m)\, e^{\frac{j2\pi mn}{N}}$$

Note that in general DFT and inverse DFT can be defined in many ways, each only differing in choice of constant C_1 and C_2

<u>DFTIDFT</u>

i.e. $\qquad X(m) = C_1\sum_{n=0}^{N-1} x_n e^{\frac{-j2\pi nm}{N}} \quad x_n = C_2\sum_{m=0}^{N-1} X(m) e^{\frac{j2\pi nm}{N}}$

The constraint in choosing the constraints is that product $C_1C_2 = {}^1\!/_N$

For Example, when

$C_1 = 1, \qquad C_2 = \dfrac{1}{N}$

$C_1 = \dfrac{2}{N} \qquad C_2 = \dfrac{1}{2}$

$C_1 = \dfrac{1}{\sqrt{N}} \qquad C_2 = \dfrac{1}{\sqrt{N}}$

Choice of $C_1 = \frac{2}{N}$ is commonly used in relaying because it simplifies phasor estimation. Phasor estimation will be discussed later. We now discuss some important properties of DFT.

Two-point DFT (N=2)

$W_2 = e^{-i\pi} = -1$, and

$$A_k = \sum_{n=0}^{1} (-1)^{kn} a_n = (-1)^{k \cdot 0} a_0 + (-1)^{k \cdot 1} a_1 = a_0 + (-1)^k a_1$$

so

$$A_0 = a_0 + a_1$$
$$A_1 = a_0 - a_1$$

Four-point DFT (N=4)

$W_4 = e^{-i\pi/2} = -i$, and

$$A_k = \sum_{n=0}^{3} (-i)^{kn} a_n = a_0 + (-i)^k a_1 + (-i)^{2k} a_2 + (-i)^{3k} a_3 = a_0 + (-i)^k a_1 + (-1)^k a_2 + i^k a_3$$

so

$$A_0 = a_0 + a_1 + a_2 + a_3$$
$$A_1 = a_0 - ia_1 - a_2 + ia_3$$
$$A_2 = a_0 - a_1 + a_2 - a_3$$
$$A_3 = a_0 + ia_1 - a_2 - ia_3$$

This can also be written as a matrix multiply:

$$\begin{bmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

More on this later.

To compute A quickly, we can pre-compute common subexpressions:

$$A_0 = (a_0 + a_2) + (a_1 + a_3)$$
$$A_1 = (a_0 - a_2) - i(a_1 - a_3)$$
$$A_2 = (a_0 + a_2) - (a_1 + a_3)$$
$$A_3 = (a_0 - a_2) + i(a_1 - a_3)$$

This saves a lot of adds. (Note that each add and multiply here is a complex (not real) operation.)

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